Transmission and Adhesive Contact Problems with Plasticity

Thesis

Ramiro Peñas Galezo

Jairo Hernández Monzón (Director)

Universidad del Norte

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Dedico esta tesis doctoral a mi hijo Miguel Ángel, de quien recibo el mayor impulso. También se la dedico a mi esposa, mis padres, y a mi círculo familiar mas cercano.

Contents

Abstract

In this thesis, problems of transmission and adhesive contact of solids with plastic deformation are investigated. Variational problems of beams, plates and deformable solids are formulated. In each case, appropriate boundary conditions are established to ensure that they are well-posed. The proof of existence and uniqueness of solutions are based on results obtained from the theory of monotonous operators.

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Frequent Symbols and Definitions

Ω	open, connected and bounded set in $\mathbb{R}^3$
ω	open, connected and bounded set in $\mathbb{R}^2$
X	Banach space
X^*	dual space of X
$W^{k,p}(\Omega)$	Sobolev space of order k and p-norm in Ω
$H^k(\Omega)$	Sobolev space $\mathtt{W}^{k,2}(\Omega)$
V, W, H	Hilbert space
$\Subset$	compact embedding
$\mathtt{I}$	identity function
$\mathtt{I}_K(x)$	indicatrix function
$A : V \rightrightarrows W$	multi-valued operator, $A : V \to 2^W$
$\mathbb{C}$	elastic tensor

$E(u) = \frac{1}{2}(\nabla u + (\nabla u)^t)$

$[P]_0 = \int_{-1}^{1} P(x_3)dx_3$

$[P]_1 = \int_{-1}^{1} x_3 P(x_3)dx_3$

$\Sigma_0(M) = \frac{2\lambda\mu}{\lambda+2\mu}(\,\mathtt{Tr}\,M)\;\mathtt{I} + 2\mu M$

$\mathtt{dev}\,M = M - \frac{1}{3}(\,\mathtt{Tr}\,M)\,\mathtt{I}$

$\partial f(x) = \Big\{x^* \in X^* : \forall y \in X, \quad f(y) - f(x) \geq \langle x^*, y - x\rangle\Big\}$

$f^*(x^*) = \underset{x\in X}{sup}\big\{\langle x^*, x\rangle - f(x)\big\}, \; x^* \in X^*$

$\mathbb{R}^{3\times 3}_{dev} := \big\{A \in \mathbb{R}^{3\times 3}_{sym} : trA = 0\big\}$

Introduction

A transmission problem consists of a system of differential equations or an abstract equation that models two or more coupled bodies in a region Γ and subject to surface and volume loads. Usually, these bodies occupy regions in space or in-plane which are defined by connected and bounded open sets. Mention may be made, among others, of the works carried out by Dautray-Lions (see p 400 [15]), Arango et. al. [3], Harutyunyan-Wolfgang [30], and Borsuk [8] for bodies in general, as well as the works of Hernandez [32] for plates and membranes with elastic deformations, and Muñoz-Portillo [42] for thermoelastic plates deformation.

A contact problem with small deformations is a system of constitutive equations or an abstract equation, which models the deformation of two or more bodies under loads. They can include effects such as damage, adhesion, memory, friction, temperature, and other dissipative responses. A classic contact damage problem of a solid located at $\Omega \subset \mathbb{R}^3$ on a foundation S considers a function $\alpha : \Omega \times [0, T] \to \mathbb{R}$ (called the fraction of damage of the material) whose evolution is typically ruled by a subdifferential inclusion. In each $x \in \Omega$, $t \in [0, T]$, α takes values between 0 and 1 (0 for total damage, $0 < \alpha < 1$ for partial damage, and 1 for no damage). There is a wide literature related to contact and damage problems in elastic, viscoelastic, and viscoplastic cases (see [2, 10, 11, 18, 19, 27, 35, 56] and the references therein). We adopt a plastic deformation model without fracture or damage, therefore we will not take into account the parameter α.

Some adhesion problems are formulated using the bonding field β and the differential inclusions introduced by Fremond [25, Ch. 7], where β is a measure of the fractional intensity of adhesion between the two solids. Like α, the parameter β takes values between 0 and 1. When $\beta = 0$, there are no active glue bonds; when $\beta = 1$, all bonds are active, and when $0 < \beta < 1$, there is a portion of active glue bonds. We can find literature related to contact problems with adhesion between a solid Ω on a foundation

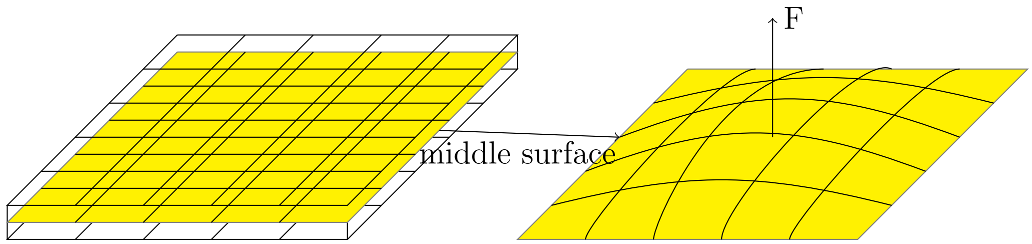

Figure 1: Deformation of the middle surface of a plate.

S in the references [2, 5, 6, 18, 27, 28, 54].

This thesis addresses transmission and contact problems between plates and deformable solids. The first models of deformation of thin plates of thickness h were deduced from the principle of virtual work applied to the Lagrangian

$$\mathscr{L} = \int_0^T [K(t) + W(t) - P(t)],$$

where K is the kinetic energy, P is the potential energy (depending on the stress tensor σ and the strain tensor e), and W is the external work applied to the plate. The differential equations that model the deformation of the middle surface of the plate were obtained from this procedure. Figure 1 shows the middle surface of a plate deformed by the action of loads. By considering different forms of admissible displacements, the Kirchhoff elastic models for small deformations, Mindlin - Timoshenko for transverse shear deformation, and the Von Karmán model for large deflections emerge. By allowing σ to be a function of temperature, the thermoelastic model is obtained, and considering σ with t-dependent coefficients, the viscoelastic model emerge (see Lagnese and Lions [34]).

Elastic models have the particularity of modeling the deformation of plates subjected to loads. By suppressing the forces that cause deformation, the system returns to its original configuration. When a material is deformed plastically by the action of external forces (which occurs when stresses exceed the elastic limit), it does not return completely to its initial state after loads are removed. This effect of retaining part of its deformation is known as hysteresis.

New forms of plate model formulation have emerged in the last decade from the Γ-convergence theory (see Dal Maso [14]). We can mention the contributions of Friesecke et al. [26], Müller [41] for elastic cases, Liero and Mielke [36] and Liero and Roche [37] for elastoplastic cases, and the models of Davoli and Mora [16] and Maggiani and Mora [38] for cases of perfect plasticity,

In this thesis, we will develop two transmission problems and two adhesive contact problems. One of the transmission problems illustrates a one-dimensional example of the deflection of two media (e.g. beams or rods) under perfect plastic deformation (see Example 2.1.1). The second transmission problem corresponds to plates subjected to loads perpendicular to their plane, which flex following the Maggiani-Mora model. This last model is formulated in the form of a Nonlinear Cauchy Problem, and it is proved that it is well-posed through results from the theory of m-accretive operators on Banach spaces. The main results are given in Theorems 2.2.8 and 2.2.9.

The two adhesive contact problems (also called delamination problems) present weak formulations of unilateral (there is no penetration from one medium to the other) and unidirectional contact (irreversible damage to the glue fibers). Both problems assume rate-independent elastoplastic deformation with hardening, problems identified according to the Mathematics Subject Classification System MSC-2020 with 74C05. As a motivation to address these problems weakly, we have the advantage of reducing the orders of derivation in the equations, and therefore, the simplification of numerical processes. The thesis presents a different formulation concerning other developments (see e.g. [24, 23, 33, 47, 49, 50, 44]) because it abstractly displays all the differential inclusions and the displacements inside and outside the contact zone in a single doubly non-linear problem. In delamination problems it is not a priority to prove the existence and uniqueness of the solutions. Its objective is to find solutions (generally called energy solutions) employing tools such as Γ-Convergence or Mosco Convergence of the Calculus of Variations. Since here we will consider weak formulations for the delamination of solids and plates, it is justified to approach the problem of the existence of solutions by thinking about further numerical developments. The solution existence proof is novel and relatively simple compared to other unidirectional adhesive contact developments in viscoelastic and perfectly plastic cases. The main results are condensed in Theorems 3.3.4 and 4.3.3. To enter into context, the irreversibility of damage in the additive interface induces a doubly nonlinear problem with unbounded operators, and its treatment has appealed to monotonicity and compactness tools (see e.g. [5, 6]). Unlike those developments, we construct a doubly nonlinear sequence of problems where one of the inclusion operators is bounded, and we use a result established by Colli [13] to guarantee existence. Later we show that the weak solution to one of the problems of the sequence is also a weak solution to the original problem.

The document has been organized as follows: Chapter 1 develops the mathematical and theoretical foundations that will be used throughout this thesis. In section 1.1 an introduction is made to monotonous operators, multivalued m-accretive operators, the Kato's theorem, and the Colli theorem. Section 1.2 develops the elastoplastic plastic models, and section 1.2.2 presents the theoretical foundations of accession.

Chapter 2 focuses on transmission problems for plate: the bidimensional case of Maggiani-Mora and one-dimensional case with perfect plasticity.

Chapter 3 is dedicated to adhesive contact problems between deformable solids with elastoplasticity and hardening.

Finally, chapter 4 addresses the problem of contact between elastoplastic plates under the Liero-Mielke model. In Chapter 3 and 4, the weak model is treated as a doubly nonlinear problem.

Chapter 1

Preliminary Theory

1.1 Some Elements of Monotone Operator Theory

In this work, for $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, $W^{k,p}(\Omega)$ will denote the usual real Sobolev space, of order k and norm p, over a domain $\Omega \subseteq \mathbb{R}^n$ (i.e. Ω open and connected). For $p = 2$ we will use the standard notation $H^k(\Omega)$, which represents the Hilbert space of order k under the inner product

$$\langle f, g \rangle_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha f(x) D^\alpha g(x) dx.$$

It is well known that the spaces $W^{k,p}(\Omega)$ are separable spaces for $k < \infty$, and reflexive for $k > 1$.

If I is any interval in $\mathbb{R}$, X a Banach space, and $1 \leq p \leq \infty$, p fixed, we define the Bochner spaces $L^p(I; X)$ by

$$L^p(I; X) = \left\{ u : I \to X : u \text{ is Bochner-measurable, and } \|u\|_{L^p(I;X)} < \infty \right\},$$

where

$$\|u\|_{L^p(I;X)} = \begin{cases} \left(\int_I \|u(t)\|_X^p \right)^{1/p}, & if \quad 1 \leq p < \infty, \\ ess \sup\limits_{t \in I} \|u(t)\|_X, & if \quad p = \infty \end{cases}.$$

If $\mathscr{D}(a, b) = \{\phi \in C^\infty(a, b; \mathbb{R}) : supp(\phi) \text{ is compact}\}$ represents the test function space on $]a, b[$ and $\mathscr{D}'(a, b; X) = \{u : \mathscr{D}(a, b) \to X,\ u \text{ is linear and continuous}\}$ the space of all X-valued distributions, then $W^{k,p}([a; b]; X)$ defines the space of all distributions $u \in \mathscr{D}'(a,b;X)$ that verify

$$u^{(j)} \in L^p(a,b;X), \qquad for\ j = 0,1,...,k,$$

where $u^{(j)}(\phi) = (-1)^j u(\phi^{(j)})$, for all $\phi \in \mathscr{D}(a,b)$. In particular,

$$H^1(a,b;X) = \left\{ u \in \mathscr{D}'(a,b;X) : u, \tfrac{du}{dt} \in L^2(a,b;X) \right\}.$$

Some results that we will apply frequently in this work are the following.

Theorem 1.1.1 (Trace operator) *Let $\Omega \subseteq \mathbb{R}^n$ a bounded domain with Lipschitz boundary, $1 \leq p < n$, $1/q = 1/p - [1/(n-1)](p-1)/p$. Then there exists exactly one bounded linear mapping $\gamma_0 : W^{1,p}(\Omega) \to L^q(\partial\Omega)$ such that $u \in C^\infty(\overline{\Omega})$ implies*

$$\gamma_0 u = u_{|\partial\Omega}.$$

Furthermore, if $k \in \mathbb{N}$ and $kp = n$, then $\gamma_0 : W^{1,p}(\Omega) \to L^q(\partial\Omega)$ is linear and bounded for every $q \geq 1$.

Proof. See [43, Th 4.2 on page 79 and Th 4.8 on page 82]. ■

Remark 1.1.2 *The mapping γ_0 is called trace operator of order zero.*

Theorem 1.1.3 (Demengel) *Let $\omega \subseteq \mathbb{R}^2$ be a bounded domain with boundary $\partial\omega$ of class C^2, with tangential and outer normal unit vectors $\tau_{\partial\omega}$ and $\eta_{\partial\omega}$, respectively. If $\hat{\sigma} \in H^2(\omega;\mathbb{R}^{2\times 2})$ and $\psi \in H^2(\omega;\mathbb{R})$, then it holds the following Green's identy:*

$$-\int_{\partial\omega} b_0(\hat{\sigma})\,\gamma_0\psi + \int_{\partial\omega} b_1(\hat{\sigma})\,\tfrac{\partial\psi}{\partial\eta_{\partial\omega}} = \int_\omega \hat{\sigma} : D^2\psi - \int_\omega \psi\ \mathtt{div}\,\mathtt{div}\ \hat{\sigma},$$

where

$$b_0(\hat{\sigma}) := \gamma_0\left(\mathtt{div}\,\hat{\sigma}\right)\cdot\eta_{\partial\omega} + \tfrac{\partial}{\partial\tau_{\partial\omega}}(\hat{\sigma}\eta_{\partial\omega}\cdot\tau_{\partial\omega}),$$
$$b_1(\hat{\sigma}) := \gamma_0\left(\hat{\sigma}\right)\,\eta_{\partial\omega}\cdot\eta_{\partial\omega},$$

and $D^2\psi$ is the Hessian matrix of ψ.

Proof. See [17, Th 2.1]. ■

Definition 1.1.4 *Let V, W be Banach spaces*

- *$A : V \to W$ is called compact if A maps bounded sets of V in relatively compact sets of W.*
- *V is embedded continuously in W ($V \hookrightarrow W$) if there is an injective and continuous mapping $\mathtt{I} : V \to W$.*
- *If $V \hookrightarrow W$ and V is a dense subset of W, we speak about a dense embedding.*
- *If $V \hookrightarrow W$ and $\mathtt{I}$ is compact, we speak about a compact embedding and use the notation $V \Subset W$.*

(See [48] for standard definitions). With the terminology above, we recall the following results for Sobolev Spaces.

Theorem 1.1.5 (Rellich-Kondrachov Compactness Theorem) *Let Ω be a domain in $\mathbb{R}^n$. Let j, m be integers, $j \geq 0$, $m \geq 1$, $1 \leq p < \infty$, and $n = mp$. Then, for $1 \leq q < \infty$,*

$$W^{j+m,p}(\Omega) \Subset W^{j,q}(\Omega).$$

In particular, $W^{2,2}(\Omega) \Subset W^{1,2}(\Omega) \Subset L^2(\Omega)$

Proof. See e.g. [1, Th 6.2] or [9, Th 9.16]). ■

Next we will define the convex functions, lower semi continuous functions, the concept of subdifferential, among other definitions (see [52] for standard definitions).

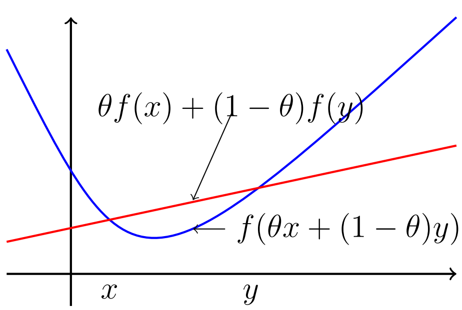

Figure 1.1: Graph of a convex function in $\mathbb{R}$.

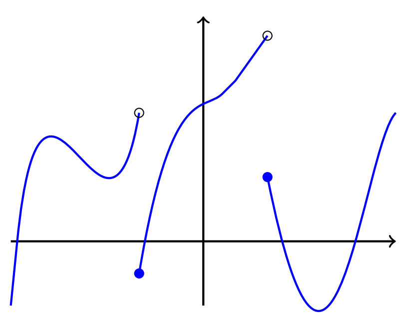

Figure 1.2: Graph of a lower semi continuous function in $\mathbb{R}$.

Definition 1.1.6 *Let X be a Banach space over the field of real numbers. A function $f : X \longrightarrow]-\infty, +\infty]$ is called*

- *Proper if $f(x) \neq \infty$ for some $x \in X$.*
- *Convex if $f(\alpha x+(1-\alpha)y) \leq \alpha f(x)+(1-\alpha)f(y)$ for every x, $y \in X$, $\alpha \in [0,1]$ (see Figure 1.1).*
- *Lower semi continuous (l.s.c.) if $f(x) \leq \liminf_{y\to x} f(y)$ for every $x \in X$ (see Figure 1.2).*
- *Subdifferentiable in x if there exists $x^* \in X^*$ such that for every $y \in X$,*

  $$\langle x^*, y-x\rangle \leq f(y) - f(x).$$

  In that case, we write $x^* \in \partial f(x)$, where $\partial f(x)$ is the subdifferential of f in x (see Figure 1.3).

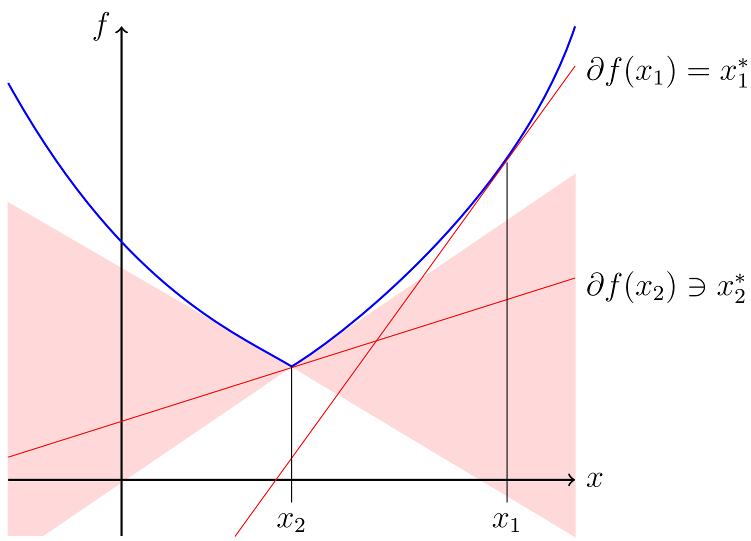

Figure 1.3: In this graph, the subdifferential of f is single-valued at x_1 and multivalued at x_2.

- $f^*: X^* \longrightarrow]-\infty, +\infty]$ *defined by* $f^*(x^*) = \sup_{x\in X} \{\langle x^*, x\rangle - f(x)\}$ *for* $x^* \in X^*$ *is called conjugate function of* f.

Example 1.1.7 *If* $K \subset X$, K *closed and convex, the indicatrix function* $\mathtt{I}_K$ *defined by*

$$\mathtt{I}_K(x) := \begin{cases} 0, & \text{if } x \in K \\ +\infty, & \text{if } x \notin K. \end{cases}$$

is convex, proper and lower semi-continuous.

Example 1.1.8 *For* $K = [a, b] \subset \mathbb{R}$, *the subdifferential of* $\mathtt{I}_K(x)$ *is*

$$\partial\, \mathtt{I}_K(x) = \begin{cases} \mathbb{R}^+ \cup \{0\}, & \text{if } x = b \\ 0, & \text{if } a < x < b \\ \mathbb{R}^- \cup \{0\}, & \text{if } x = a \end{cases}.$$

Figure 1.4 shows $\partial\, \mathtt{I}_K(x)$ *when* $K = [0, 1]$.

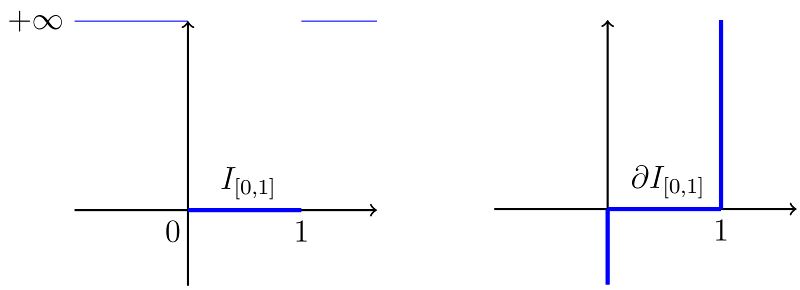

Figure 1.4: On the left the graph of $\mathtt{I}_{[0,1]}(x)$, on the right the graph of $\partial\, \mathtt{I}_{[0,1]}(x)$ in $\mathbb{R}$.

The following result gives sufficient conditions for a function defined in a reflexive Banach space and with values in $(-\infty, \infty]$, to reach a minimum.

Theorem 1.1.9 *Let* K *be a convex, closed and non-empty subset in a reflexive Banach space* X, *and let* $F : X \longrightarrow (-\infty, +\infty]$ *be convex and lower semi-continuous. If*

$$\lim_{x\in K, \|x\|\to\infty} F(x) = +\infty,$$

then there is $x_m \in K$ *which is a minimum point of* F *in* K *(i.e.* $F(x_m) \le F(y)$ *for all* $y \in K$*).*

Proof. See [52, Prop. 1.4, Ch IV]. ■

Definition 1.1.10 *Let V, W be linear spaces. A multivalued operator A of V in W is a relation from V to W, that is, a subset of $V \times W$ (see [52, pp 158] or [48, pp 8]). Such operator or mapping can be represented by $A : V \rightrightarrows W$ or $A : V \to 2^W$, where 2^W is the collections of all subsets of W, and its images be defined by $y \in A(x)$ if $(x, y) \in A$. The domain and the range of the operator are defined by $D(A) = \{x : (x, y) \in A \text{ for some } y \in W\}$ and $Rg(A) = \{y : (x, y) \in A \text{ for some } x \in V\}$ respectively.*

The models that will be considered in this work are formulated in terms of monotone or accretive operators.

Definition 1.1.11 *Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space. A multi-valued operator $A : H \rightrightarrows H$ is* accretive *or* monotone *if given $(x, y), (\hat{x}, \hat{y}) \in A$, is verified*

$$\langle \hat{x} - x, \hat{y} - y\rangle_H \geq 0.$$

A is m-accretive *if A is accretive and $Rg(\mathtt{I} + \epsilon A) = H$ for some $\epsilon > 0$ (see [52, pp 158]).*

Theorem 1.1.12 *Let H be a Hilbert space, and $\varphi : H \longrightarrow (-\infty, +\infty]$ convex, proper and lower-semi-continuous. Then*

1. *If $\varphi : H \longrightarrow (-\infty, +\infty]$ is continuous in $x \in H$, then $\partial\varphi(x) \neq \emptyset$.*
2. *$\partial\varphi : H \rightrightarrows H$ is acretive (monotone)*
3. *The range of $I + \partial\varphi$ is H, i.e. $Rg(I + \partial\varphi) = H$.*

Proof. *1.* See [21, Prop 5.2 Ch I].
2. See [52, Remark p.157 Ch IV].
3. See [52, Prop. 1.5 Ch IV]. ■

Remark 1.1.13 *If $H = L^2(\Omega; \mathbb{R})$, $\Omega \subset \mathbb{R}$, then each functional $\mathtt{I}_K : \mathbb{R} \to]-\infty, +\infty]$ can be extended over H to a functional $\mathtt{I}_{\bar{K}}$, where*

$$\bar{K} := \left\{\beta \in L^2(\Omega; \mathbb{R}) : \beta(x) \in K \text{ for a.e. } x \in \Omega\right\}.$$

In fact,

$$\mathtt{I}_{\bar{K}}(\beta) := \begin{cases} 0, & \text{if } \beta \in \bar{K} \\ +\infty, & \text{if } \beta \notin \bar{K} \end{cases} = \int_\Omega \mathtt{I}_K(\beta(x))dx.$$

On the other hand, if $\varrho \in H$ *and* $\varrho(x) \in \partial\,\mathtt{I}_K(\beta(x))$ *for a.e.* $x \in \Omega$*, then the functional* $\varrho^* := \langle \varrho, \cdot \rangle \in H^*$ *verifies,*

$$\begin{aligned} \varrho^*(\alpha - \beta) &= \langle \varrho, \alpha - \beta \rangle = \int_\Omega \varrho(x)\left(\alpha(x) - \beta(x)\right)dx \\ &\leq \int_\Omega \left[\mathtt{I}_K(\alpha(x)) - \mathtt{I}_K(\beta(x))\right]dx = \mathtt{I}_{\bar{K}}(\alpha) - \mathtt{I}_{\bar{K}}(\beta), \quad \text{for all } \alpha \in H, \end{aligned}$$

and therefore, $\varrho^* \in \partial_\beta\,\mathtt{I}_{\bar{K}}(\beta)$ *(i.e. a local condition defined in terms of* $\partial\,\mathtt{I}_K(\beta(x))$ *induces a global condition). Henceforth we will symbolize* $\mathtt{I}_K(\beta)$ *instead of* $\mathtt{I}_{\bar{K}}(\beta)$*, and* $\partial\,\mathtt{I}_K(\beta)$ *instead of* $\partial_\beta\,\mathtt{I}_{\bar{K}}(\beta)$*.*

1.1.1 The Nonlinear Cauchy Problem Associated to a Multivalued Operator

Let H be a Hilbert space and $A : H \rightrightarrows H$ be a multivalued operator. We will consider the following abstract nonlinear Cauchy problem associated with the operator A.

Problem 1.1.14 *Given* $T > 0$, $f \in L^1(0,T;H)$, $u_0 \in D(A)$*, find* $u \in C([0,T];H)$ *such that* $\frac{du}{dt}$ *exists in a distributional sense,* $\frac{du}{dt}(t) \in H$ *for* $t \in (0,T]$*, and*

$$\begin{cases} \frac{d}{dt}u(t) + Au(t) \ni f(t), \ t \in (0,T], \\ u(0) = u_0 \end{cases} \tag{1.1}$$

The following theorem gives sufficient conditions for the existence and uniqueness of a kind of solutions of (1.1).

Theorem 1.1.15 *Let* A *be a m-accretive operator in a reflexive Banach space* H*. For each* $u_0 \in D(A)$ *and* $f \in W^{1,1}(0,T;H)$*, there is a unique* $u \in W^{1,\infty}(0,T;H)$ *that satisfies the Nonlinear Cauchy Problem* (1.1) *at a.e.* $t \in [0,T]$*. Such solution is called a strong solution of Problem* (1.1)*.*

Proof. see e.g. [4, Th 4.5]. ∎

The solution (if it exists) of an evolution problem of type (1.1) has representation in terms of the elements of a semigroup of operators. The following definitions of semigroups are standard and can be consulted for example in Pazy [45].

Definition 1.1.16 *Let X be a Banach space. A family $\{T(t)\}_{t\geq 0}$ of operators on X is a semigroup on X, if it verifies*

$$\begin{aligned} &1. \quad T(t+s) = T(t)T(s),\ t \geq 0, s \geq 0 \\ &2. \quad T(0) = I \end{aligned}$$

Definition 1.1.17 *Let X be a Banach space. A family $\{T(t)\}_{t\geq 0}$ of linear operators bounded on X is a C_0 semigroup on X if verifies Definition 1.1.16 and*

$$\lim_{t\to 0^+} \|T(t)u - u\|_X = 0 \text{ for each } u \in X.$$

If $\lim_{t\to 0^+} \|T(t) - I\|_{\mathscr{L}(X)} = 0$, we say that the semigroup is uniformly continuous.

Definition 1.1.18 *The operator $A : D(A) \to X$ defined by*

$$D(A) = \left\{ x \in X : \frac{T(h)x - x}{h} \text{ converges on } X \text{ when } h \to 0^+ \right\}$$

$$Ax = \lim_{h\to 0^+} \frac{T(h)x - x}{h}$$

(if it exists) is called the infinitesimal generator of the semigroup $T(t)$.

We can obtain the representation of a uniformly continuous semigroup from its infinitesimal generator when it is linear and bounded.

Theorem 1.1.19 *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator. In such a case,*

$$T(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \tag{1.2}$$

Proof. See [45, Th 1.2.]. ■

Problem 1.1.20 *If A is the infinitesimal generator of a C_0 semigroup on a Banach space X, the non-homogeneous initial value problem associated to A is defined by*

$$\left.\begin{array}{c} \text{Given } T>0,\ u_0 \in X,\ \text{ and } f:[0,T[\to X,\ \text{find}\, u:[0,T[\to X \text{ such that} \\ \dfrac{du}{dt}(t) = Au(t) + f(t),\ t \in]0,T[\\ u(0) = u_0. \end{array}\right\} \tag{1.3}$$

Definition 1.1.21 *A (classical) solution of* (1.3) *is a function $u \in C([0,T[;X) \cap C^1(]0,T[;X)$, such that $u(t) \in D(A)$ for $0 < t < T$, the differential equation in* (1.3) *is satisfied on $]0,T[$, and $u(0) = u_0$.*

Results of existence and uniqueness of solutions can be found in [45].

Theorem 1.1.22 *Let $T(t)$ be the semigroup generated by A, $f \in L^1(0,T;X)$. Then for all $u_0 \in X$ the initial value problem* (1.3) *has at most a solution $u(t)$. Such a solution if it exists is given by the Duhamel formula*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds. \tag{1.4}$$

Proof. See [45, Ch 4, Cor 2.2]. ■

Definition 1.1.23 *If $f \in L^1(0,T;X)$, the continuous function given by* (1.4) *is called a mild solution of problem* (1.3).

1.1.2 Doubly Nonlinear Problems

The abstract differential inclusion related to two of the models proposed in this thesis, has the representation

$$\partial\varphi(\dot{z}) + \partial\psi(z) \ni f, \tag{1.5}$$

where $\dot{z} = dz/dt$, f is a load applied to the system, φ and ψ are functional on a space that we will specify later, and ∂ represents the subdifferential applied to a functional[1]. Results on existence and uniqueness of solutions are referenced in [12, 13, 48], and the references therein. In particular, we will make use of one of the results of Colli [13], where $\partial\varphi$ and $\partial\psi$ are monotone operators. The theorem that guarantees the existence of solutions of (1.5) is enunciated in Theorem 1.1.25.

[1] $x^* \in \partial\varphi(x)$ if $x^* \in V^*$ and $\langle x^*, y-x\rangle \leq \varphi(y) - \varphi(x)$ for every $y \in V$.

Definition 1.1.24 *Let V be a Banach space and $A : V \rightrightarrows V^*$ a multivalued operator. A is* monotone *if for all (x_1, y_1), $(x_2, y_2) \in A$, $\Big\langle\, y_1 - y_2, x_1 - x_2 \,\Big\rangle \geq 0$. A is* strongly monotone *if there exists $C > 0$ such that for any (x_1, y_1), $(x_2, y_2) \in A$,*

$$\Big\langle\, y_1 - y_2, x_1 - x_2 \,\Big\rangle \geq C \left\| x_1 - x_2 \right\|_V^2 .$$

Let V, W be Banach spaces with $V \hookrightarrow W$, V dense in W and $V \Subset W$. Furthermore let W be reflexive and *strictly convex*[2]. We will consider the problem

$$\begin{aligned} A\left(\overset{\bullet}{z}(t)\right) + B(z(t)) &\ni f(t) \qquad \text{for a.e. } t \in]0, T[\\ & z(0) \in V, \end{aligned}$$

where $A : W \rightrightarrows W^*$, $B : V \rightrightarrows V^*$, and $f :]0, T[\to W^*$.

Theorem 1.1.25 (Colli) *Let V, W such that, $V \Subset W$, $V \hookrightarrow W$, V dense in W, and let φ, $\psi : W \to]-\infty, +\infty]$ proper*[3]*, convex, and lower semi-continuous (l.s.c.) functions such that*

1. *$\partial\varphi : W \rightrightarrows W^*$ is bounded (i.e. maps bounded sets into bounded sets),*
2. *$\partial\psi : W \rightrightarrows W^*$ is strongly monotone in V,*
3. *$f \in L^1(0, T; W^*) \cap H^1(0, T; V^*)$,*
4. *$z_0 \in V$ and there exists $v_0 \in \partial\psi(z_0) \subset V^*$,*
5. *$f(0) - v_0 \in D(\varphi^*)$, where φ^* is the conjugate function*[4] *of φ.*

Then there exists a triple $z \in H^1(0, T; V)$, $w \in L^\infty(0, T; W^*)$, $v \in L^1(0, T; W^*) \cap L^\infty(0, T; V)$ satisfing

$$\begin{aligned} w(t) + v(t) &= f(t), \\ w(t) &\in \partial\varphi(\overset{\bullet}{z}(t)), \\ v(t) &\in \partial\psi(z(t)) \quad \textit{for a.e. } t \in]0, T[, \\ z(0) &= z_0. \end{aligned}$$

[2] V strictly convex if the sphere in V does not contain any line segment.
[3] $f(x) \neq \infty$ for some $x \in V$.
[4] $\varphi^*(x^*) = \sup\limits_{x \in V} \left\{ \langle x^*, x \rangle - \varphi(x) \right\}$, $x^* \in V^*$, and $y \in D(\varphi^*)$ if $\varphi^*(y) < \infty$.

Proof. See [13, Th 3]. ■

1.2 About Elastoplasticity and Adhesion Models

1.2.1 Linearized Elastoplasticity in Rate-Independent System

Rate-independent systems comprise physical processes that manifest instantaneously or on very short time scales. From a mathematical approach, these processes are described by constitutive equations which are invariant under a change of time scale, particularly under a strictly monotone reparametrization on the variable t. Examples of rate-independent processes include the phenomena of elastoplasticity, delamination damage, fracture propagation, and ferroelectricity, among others. For a more detailed interpretation of this process, the reader can consult [39], [40].

The elastoplastic contact model that we will address is defined by a momentum equation and a plastic flow rule. The plastic flow rule is a differential inclusion as a function of the plastic and strain tensor, which relates the subdifferentials of an energy functional and a dissipation potential; the latter is expressed as a function of the speed of the plastic tensor but not of the rate of deformation.

To define the contact model, we consider two domains (open and connected sets) $\Omega_i \subset \mathbb{R}^3$, $i = 1, 2$, with boundary $\partial\Omega_i$ of class C^1, and let $\Gamma_{0_i} \subset \partial\Omega_i$ with $\mathtt{meas}(\Gamma_{0_i}) > 0$, $i = 1, 2$, and let $H^1_{\Gamma_{0_i}}(\Omega_i) := \left\{u^i \in H^1(\Omega_i)^3 : u^i|_{\Gamma_{0_i}} = 0\right\}$.

The elastoplastic properties of each body Ω_i are prescribed by the stored energy density $\mathtt{W}_i(e, p)$, and the potential of dissipation $\mathtt{R}_i(e)$ (see [53]), where:

- $e(u) := \frac{1}{2}\left(\nabla u + (\nabla u)^{\intercal}\right)$ is the elastic component and p the plastic component of the strain tensor
- $e(u) \in L^2(\Omega_i)^{3\times3}_{sym} := L^2(\Omega_i; \mathbb{R}^{3\times3}_{sym})$,
- $p \in L^2(\Omega_i)^{3\times3}_{dev} := L^2(\Omega_i; \mathbb{R}^{3\times3}_{dev})$, $\quad \mathbb{R}^{3\times3}_{dev} = \left\{A \in \mathbb{R}^{3\times3}_{sym} :\ \mathrm{tr}(A) = 0\right\}$,
- $\mathtt{W}_i : L^2(\Omega_i)^{3\times3}_{sym} \times L^2(\Omega_i)^{3\times3}_{dev} \to \mathbb{R}$ is continuous and coercive,
- $\mathtt{R}_i : L^2(\Omega_i; \mathbb{R}^{3\times3}_{dev}) \to \mathbb{R}$ is continuous, convex and 1-homogeneous[5].

[5] i.e., $\mathtt{R}(\lambda p) = \lambda\mathtt{R}(p)$ for all $\lambda > 0$, which guarantees that the material response is rate-independent.

The solution of a rate-independent linearized elastoplastic problem has to solve the *momentum equation*

$$-\,\mathtt{div}\,(\partial_e \mathtt{W}_i(e(u^i), p^i)) = f^i \quad in \quad \Omega_i, \tag{1.6}$$

and a differential inclusion called *plastic flow rule*

$$0 \in \partial\,\mathtt{R}_i(\dot{p}^i) + \partial_p \mathtt{W}_i(e(u^i), p^i) \quad in \quad \Omega_i, \tag{1.7}$$

where $\partial\,\mathtt{R}_i(p)$ is the subdifferential of $\mathtt{R}_i$ in p, and f^i it is a density of force per unit of volume. Particularly we will consider as stored energy density to the quadratic functional[6]

$$\begin{aligned} \mathtt{W}_i(e,p) &= \left\langle \tfrac{1}{2}\mathcal{C}(e-p),(e-p)\right\rangle + \tfrac{h_i}{2}\|\,p\,\|^{\,2}, \\ &= \tfrac{\lambda_i}{2}(\mathtt{tr}\,e)^2 + \mu_i\,\|\,e-p\,\|^{\,2} + \tfrac{h_i}{2}\|\,p\,\|^{\,2}, \quad i=1,2, \end{aligned} \tag{1.8}$$

where $\mathcal{C}(e) = \frac{\lambda_i}{2}(\mathtt{tr}\,e)\mathtt{I}_{\mathtt{d}} + 2\mu e$, λ_i, μ_i are the coefficients of Lamé and h_i a measure for the kinematic hardening. In this particular case, the stress tensor is determined by the derivative of the energy functional with respect to the deformation tensor: $\sigma = \partial_e \mathtt{W}(e(u),p)$. On the other hand, dissipation potential is considered as

$$\mathtt{R}_i(p) = \sigma_{yield(\Omega_i)}\,\|\,p\,\|\,,$$

where σ_{yield} is the yield stress. According to (1.8), we have

$$\sigma = \lambda(tr\,e)\,\mathtt{I}_{\mathtt{d}} + 2\mu\,(e-p) \in \mathbb{R}^{3\times 3}_{sym},$$

and

$$\partial_p \mathtt{W}(e(u),p) = -2\mu\,(e-p) + hp.$$

On the system (1.6)-(1.7), we will assign the boundary conditions

$$\left.\begin{aligned} \sigma^1\,\mathtt{N}_1 &= \partial_e \mathtt{W}_1(e(u^1),p^1)\,\mathtt{N}_1 = g^1, \text{ on } \partial\Omega_1\backslash\Gamma \\ \sigma^2\,\mathtt{N}_2 &= \partial_e \mathtt{W}_2(e(u^2),p^2)\,\mathtt{N}_2 = g^2, \text{ on } \partial\Omega_2\backslash\Gamma \end{aligned}\right\} \tag{1.9}$$

where $\mathtt{N}_i$ denotes the normal external vector to $\partial\Omega_i$, g^i is a force density per unit area

[6] in the isotropic and homogeneous case.

and $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$.

1.2.2 Adhesion

A contact problem with small deformations is a system of constitutive equations or abstract equations, which models the deformation of two or more bodies under load. They can include effects such as damage, adhesion, memory, friction, temperature, and other dissipative responses. The classic adhesive contact models such as the one discussed in this work are idealized cases that assume cohesive zones independent of time, and that generally introduce inconsistencies in the model (see [31] for a more detailed discussion); However, the analysis in [31] is outside the theoretical scope of this work as it is related to the thermodynamics associated with the boundary (see [55]), and instead we will consider the standard approach of [25] on adhesive contact.

Suppose that Ω_1 and Ω_2 are glued in a common region of contact Γ of class C^1. In addition to the variable e, p introduced in section 1.2.1, we consider the variable $\beta : \Gamma \to [0,1]$ which models the evolution of the surface fraction with active glue fibers, which break or mend by microscopic motions (solid glue is assumed and for such irreversibility of the break). When $\beta = 0$ there is no active glue bonds, when $\beta = 1$ all bonds are active, when $0 < \beta < 1$, there is a portion of active glue bonds.

Because of the conditions imposed on the border of solids, trace theorem extends the displacement field $u(x)$ with $x \in \Omega_i$ to each of the points $x \in \partial\Omega_i$, in particular, it can be extended over Γ. It will be denoted by $\| u^2_{|\Gamma} - u^1_{|\Gamma} \|$ the gap on the contact surface Γ, where $u^i_{|\Gamma}$ is the small displacement of the solids Ω_i on Γ at the macroscopic level. For the sake of simplicity, it neglects the thermal phenomena and excludes the temperature as a state quantity. Also, no work involving microscopic motions is provided to the system, i.e., there are no chemical, radiactive, optical, or electrical actions. The differential inclusions used by Fremond [25] for the adhesion problem are

$$c_s\dot{\beta} - k_s\Delta_s\beta + \partial\, \mathtt{I}_{[0,1]}(\beta) + \partial\, \mathtt{I}_{\text{-}}(\dot{\beta}) \ni \omega_s - \tfrac{k}{2}\| u^2_{|\Gamma} - u^1_{|\Gamma} \|^2, \text{ on } \Gamma \tag{1.10}$$

$$\left.\begin{aligned} &\sigma^1\, \mathtt{N}_1 - k\beta(u^2_{|\Gamma} - u^1_{|\Gamma}) - \partial\, \mathtt{I}_{\text{-}}((u^2_{|\Gamma} - u^1_{|\Gamma})\cdot \mathtt{N}_2)\, \mathtt{N}_2 \ni 0, \\ &\sigma^2\, \mathtt{N}_2 + k\beta(u^2_{|\Gamma} - u^1_{|\Gamma}) + \partial\, \mathtt{I}_{\text{-}}((u^2_{|\Gamma} - u^1_{|\Gamma})\cdot \mathtt{N}_2)\, \mathtt{N}_2 \ni 0, \end{aligned}\right\} \text{ on } \Gamma \tag{1.11}$$

where:

- the parameter k_s measures the intensity of microscopic interactions,
- k is an elastic constant of the adhesive material,
- c_s is the viscosity coefficient of the glue,
- $\mathtt{I}_A(x) = \begin{cases} 0, & \text{if } x \in A \\ +\infty, & \text{if } x \notin A \end{cases}$ is the indicator function, $\mathtt{I}_{\text{-}}(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ +\infty, & \text{if } x > 0 \end{cases}$,
- $\mathtt{I}_{\text{-}}((u^2_{|\Gamma} - u^1_{|\Gamma}) \cdot \mathtt{N}_2)$ is an impenetrability constraint of both solids (unilateral contact)
- $\mathtt{I}_{\text{-}}(\dot{\beta})$ characterizes the irreversible behavior of solid glues (uni-directional contact)
- ω_s is the energy of Dupré which consists of the work required to separate two adhered bodies.

In case of considering reversible adhesion, the relation $\partial\, \mathtt{I}_{\text{-}}(\dot{\beta})$ is removed from the equation (1.10). The boundary and initial conditions for β are

$$k \frac{\partial \beta}{\partial\, \mathtt{n}_s} = 0, \text{ on } \partial\Gamma \tag{1.12}$$

$$\beta(x, 0) = \beta_0(x), \text{ on } \Gamma \tag{1.13}$$

where $\mathtt{n}_s$ denotes the normal vector exterior to $\partial\Gamma$.

Chapter 2

Transmission Problems with Perfect Plasticity

This chapter develops the transmission problem between two homogeneous bodies with perfect plastic deformation. The first section illustrates a one-dimensional case of transmission between two bodies occupying two mediums ω_1 and ω_2 ($\omega_1, \omega_2 \subset \mathbb{R}$), deforming plastically under the relations

$$\left.\begin{aligned} v_{i,t} - k_i\sigma_{i,x} &= f_i(x,t) \\ \sigma_{i,t} - v_{i,x} + \partial_\sigma \mathtt{I}_{K_i}(\sigma_i) &\ni 0 \end{aligned}\right\} \quad in \quad \omega_i \times [0,T], \quad i = 1,2,$$

where k_i is a constant that depends on the material, K_i is the elastic domain, and v_i, σ_i, f_i are respectively the velocities, the stress tensor and the applied loads, in a direction perpendicular to the medium i, $i = 1,2$ (see [51]). This example will be solved by interpreting the problem as a nonlinear Cauchy problem and applying the Duhamel formula (1.4) and the representation of semigroups $T(t)$ in the form of a series $T(t) = \sum \frac{t^n A^n}{n!}$. Although the representation of the semigroup $T(t)$ depends on the bounding of the operator A (see [45, Th 1.2]), and A is a differential operator and therefore unbounded, we can obtain a representation $T(t)\binom{v}{\sigma}$ if v, σ are polynomial functions that can be vanished by A^n for some $n \in \mathbb{N}$.

The second section addresses the problem of transmission of perfectly plastic plates and the existence and uniqueness of solutions for the problem are proved.

2.1 A One-Dimensional Case

Example 2.1.1 *Let us assume that two bodies occupying* $\omega_1 =]0,1[$ *and* $\omega_2 =]1,2[$ *are in contact at* $\gamma = \{1\}$, *and that a load per unit length* $f(x,t) = 4t$ *acts on them. Let us also assume that the evolution equations acting on each* ω_i *are:*

$$\left.\begin{aligned} v_{1,t} - 2\sigma_{1,x} &= 2t \\ \sigma_{1,t} - v_{1,x} + \partial_\sigma \mathtt{I}_{K_1}(\sigma_1) &\ni 0 \end{aligned}\right\} \quad in \quad \omega_1 =]0,1[, \tag{2.1}$$

$$\left.\begin{aligned} v_{2,t} - \sigma_{2,x} &= 4t \\ \sigma_{2,t} - v_{2,x} + \partial_\sigma \mathtt{I}_{K_2}(\sigma_2) &\ni 1 \end{aligned}\right\} \quad in \quad \omega_2 =]1,2[\tag{2.2}$$

(i.e. $k_1 = 2$, $k_2 = 1$*), with initial conditions*

$$\begin{aligned} \begin{pmatrix} v_1 \\ \sigma_1 \end{pmatrix}(x,0) &= \begin{pmatrix} v_1 \\ \sigma_1 \end{pmatrix}(x) = \begin{pmatrix} -x^2 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} v_2 \\ \sigma_2 \end{pmatrix}(x,0) &= \begin{pmatrix} v_2 \\ \sigma_2 \end{pmatrix}(x) = \begin{pmatrix} -2x^2 + x \\ 0 \end{pmatrix}, \end{aligned}$$

and transmission condition

$$\lim_{x\to 1^-} \begin{pmatrix} v_1 \\ \sigma_1 \end{pmatrix}(x,t) = \lim_{x\to 1^+} \begin{pmatrix} v_2 \\ \sigma_2 \end{pmatrix}(x,t).$$

Finally we consider that the body occupying the medium ω_1 *is clamped at* $x = 0$ *and* ω_2 *is free at* $x = 2$. *(i.e., the boundary conditions* $v_1(0,t) = 0$, $v_1(1,t) = v_2(1,t)$, *and* $\partial_x v_1(0,t) = 0$ *for all* t*). Combining the Duhamel formula* (1.4) *and the representation* (1.2) *for semigroups, one can obtain*

$$\begin{pmatrix} v_1 \\ \sigma_1 \end{pmatrix} = \begin{pmatrix} -x^2 \\ -2xt - \mathtt{I}_{K_1}(t) \end{pmatrix} \quad and \quad \begin{pmatrix} v_2 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} -2x^2 + x \\ -(4x-2)t - \mathtt{I}_{K_2}(t) \end{pmatrix}, \tag{2.3}$$

where

$$\frac{d}{dt}\begin{pmatrix} v_1 & v_2 \\ \sigma_1 & \sigma_2 \end{pmatrix} + (A_1 + A_2)\begin{pmatrix} v_1 & v_2 \\ \sigma_1 & \sigma_2 \end{pmatrix} \ni \begin{pmatrix} f_{v_1} & f_{v_2} \\ f_{\sigma_1} & f_{\sigma_2} \end{pmatrix}$$

is the nonlinear Cauchy problem associated with the problem, and

$$A_1\begin{pmatrix} v_1 & v_2 \\ \sigma_1 & \sigma_2 \end{pmatrix} \ : \ = \left(\begin{pmatrix} 0 & -2\partial_x \\ -\partial_x & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ \sigma_1 \end{pmatrix}, \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}\begin{pmatrix} v_2 \\ \sigma_2 \end{pmatrix}\right),$$

$$A_2\begin{pmatrix} v_1 & v_2 \\ \sigma_1 & \sigma_2 \end{pmatrix} \ : \ = \begin{pmatrix} 0 & 0 \\ \partial_\sigma \mathtt{I}_{K_1}(\sigma_1) & \partial_\sigma \mathtt{I}_{K_2}(\sigma_2) \end{pmatrix},$$

$$\begin{pmatrix} f_{v_1} & f_{v_2} \\ f_{\sigma_1} & f_{\sigma_2} \end{pmatrix} \ : \ = \begin{pmatrix} 2t & 4t \\ 0 & 1 \end{pmatrix}.$$

The calculations have been omitted due to the length of the details, however, the reader can easily verify that (2.3) *satisfies* (2.1)-(2.2)*. If we now consider* $K_1 = W^{1,2}(\omega_1)$, $K_2 = W^{1,2}(\omega_2)$*, then* $\mathtt{I}_{K_1}(\sigma) = \mathtt{I}_{K_2}(\sigma) = 0$*, and instead of a perfectly plastic model we get an elastic model:*

$$\begin{pmatrix} v_1 \\ \sigma_1 \end{pmatrix} = \begin{pmatrix} -x^2 \\ -2xt \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_2 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} -2x^2 + x \\ -(4x-2)t \end{pmatrix}. \tag{2.4}$$

Since (2.4) *is defined in terms of velocity, we can assign initial and transmission conditions on each* v_i *in* (2.4)*, to obtain the evolution as a function of the vertical displacement* y_i *of the section* ω_i, $i = 1, 2$*. Figure 2.1 illustrates the case with initial and transmission conditions:* $\quad y_1(0,0) = 0, \quad y_2(1,t) = y_1(1,t), \quad t \geq 0.$

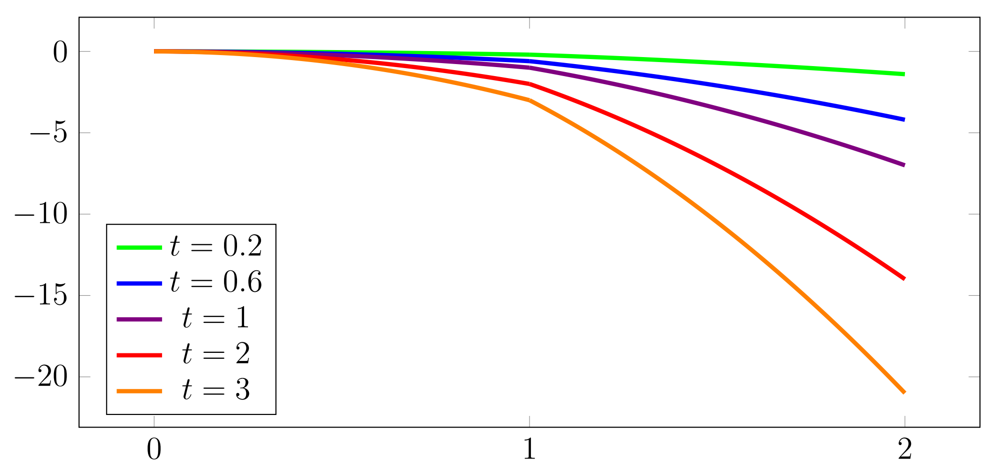

Figure 2.1: Evolution of the transmission.

2.2 Perfect Plastic Plates

This section formulates the transmission model of perfectly plastic plates under the action of forces perpendicular to its plane. The system of equations that we will use is in general, for $t \in (0, T]$, $T > 0$, given by:

$$\ddot{u}(t) - \tfrac{1}{12}\,\mathtt{div}\,\mathtt{div}\,(\mathbb{C}e(t)) = f(t) \quad in \quad \omega, \tag{2.5}$$

$$D^2 u = -(e + p) \quad in \quad \omega, \tag{2.6}$$

$$\dot{p} \in \partial\, \mathtt{I}_K(\sigma), \tag{2.7}$$

where $\omega \subset \mathbb{R}^2$ is a domain (that is, an open, connected, and bounded set), K is a convex and compact set in the space of 2×2 symmetric matrices, u is the vertical displacement of the plate, $e \in \mathbb{R}^{2\times 2}_{sym}$ is the strain field, $\mathbb{C}$ is the elastic tensor, $\mathbb{C}e = \sigma \in \mathbb{R}^{2\times 2}_{sym}$ is the stress field, $\mathtt{div}\,\mathtt{div}\,(\mathbb{C}e(t)) := \sum_{i,j=1}^{2} \partial_{ij}\mathbb{C}e_{ij}(t)$, and p is the plastic field. The relation $\sigma = \mathbb{C}e$ reads as follows:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{22} \end{pmatrix} = \frac{E}{1+\nu} \begin{pmatrix} \frac{1-\nu}{1-2\nu} & & & \frac{\nu}{1-2\nu} \\ & 1 & & \\ & & 1 & \\ \frac{\nu}{1-2\nu} & & & \frac{1-\nu}{1-2\nu} \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{pmatrix},$$

where E is Young's modulus and ν is Poisson's coefficient of the plate. Equations (2.5)-(2.7) have been taken from the work of Maggiani-Mora [38], which constitutes a more general model of perfect plastic deformation derived by Γ-convergence. The plate configuration and the assumptions for their model are:

- The plate is composed of a homogeneous and isotropic material,
- $\Omega = \omega \times [-\frac{h}{2}, \frac{h}{2}]$ has a C^2-boundary, and plate displacements are prescribed on a part of that boundary,
- The plate is subjected to purely vertical body loads f,
- Admissible plate deformations are in a Kirchoff-Love space (see [34]),

- The elastic behavior of the deformation is linear and follows the Prandtl-Reuss law of perfect plasticity.

We will define a solution space that is consistent with the boundary conditions, and we will formulate the transmission problem in weak form as a non-linear Cauchy problem. We will show that the associated operator is m-accretive and we will guarantee the existence and uniqueness of weak solutions through Theorem 1.1.15.

2.2.1 Configuration of the Transmission Model

Two thin plates Ω_1 and Ω_2 of the same thickness that project regions ω_1 and ω_2 in the plane, are joined in $\Gamma_c = \overline{\Omega_1} \cap \overline{\Omega_2}$. It is assumed that $\partial(\omega_1 \cup \omega_2)\backslash\gamma_c$ and γ_c are of class C^2, and that both Ω_1 and Ω_2 are made of the same or different materials, each of homogeneous and isotropic nature. The elastic behavior of each plate is linear and their respective plastic responses follow the Prandtl-Reuss law on perfect plasticity (see Duvaut-Lions [20]).

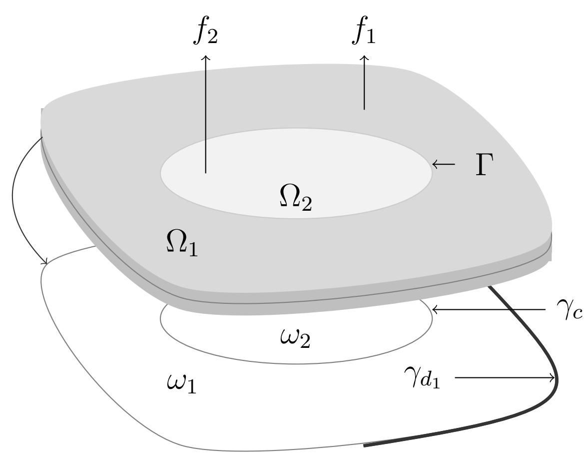

Figure 2.2: In this particular configuration only one of the plates has a prescribed displacement on γ_d. ω_1 is clamped on $\gamma_{d_1} \neq \emptyset$, $\gamma_{d_2} = \emptyset$, γ_c is the contact zone, and $\gamma_{n_i} = \partial\omega_i\backslash(\gamma_c \cup \gamma_{d_i}),\ i = 1, 2.$

We must find an L^1 integrable triplet $(u, v, e) : [0, T] \longrightarrow H$, where H is a suitable Hilbert space, with $v = \dot{u}$. Because v is L^1 integrable in the interval $[0, T]$, the displacement u can be calculated after determining the velocity v. Taking this in account, we consider the evolution problem

$$\left.\begin{array}{l}\frac{d}{dt}v_i - \frac{1}{12}\,\mathtt{div}\,\mathtt{div}\,(\mathbb{C}e_i) = f_i \\ \frac{d}{dt}e_i + D^2 v_i + \partial \mathtt{I}_{K_i}(\mathbb{C}e_i) \ni 0\end{array}\right\} \text{ in } \omega_i,\ i=1,2, \quad t\in(0,t], \tag{2.8}$$

with transmission conditions

$$\left.\begin{array}{c} v_1 = v_2\,, \qquad \nabla v_1 = \nabla v_2\,, \qquad e_1 = e_2 \\ (\mathtt{div}\,\mathbb{C}e_1)\cdot\eta_{\partial\omega_1} = -(\mathtt{div}\,\mathbb{C}e_2)\cdot\eta_{\partial\omega_2} \\ \frac{\partial}{\partial\tau_{\partial\omega_1}}((\mathbb{C}e_1)\,\tau_{\partial\omega_1}\cdot\eta_{\partial\omega_1}) = -\frac{\partial}{\partial\tau_{\partial\omega_2}}((\mathbb{C}e_2)\,\tau_{\partial\omega_2}\cdot\eta_{\partial\omega_2})\end{array}\right\} \text{ on } \gamma_c \tag{2.9}$$

and boundary conditions

$$\left.\begin{array}{l} \left.v_i = 0, \qquad \frac{\partial v_i}{\partial \eta_{\partial\omega}} = 0, \qquad i=1,2\right\} \quad \text{on } \gamma_{d_i} \\ \left.\begin{array}{c}(\mathtt{div}\ \mathbb{C}e_i)\cdot\eta_{\partial\omega} + \frac{\partial}{\partial\tau_{\partial\omega}}((\mathbb{C}e_i)\,\tau_{\partial\omega}\cdot\eta_{\partial\omega}) = 0, \\ (\mathbb{C}e_i)\,\eta_{\partial\omega}\cdot\eta_{\partial\omega} = 0,\end{array}\right\} \begin{array}{c}\text{on } \gamma_{n_i} = \partial\omega_i\backslash(\gamma_c\cup\gamma_{d_i}), \\ i=1,2.\end{array}\end{array}\right\}. \tag{2.10}$$

Furthermore we give the following initial conditions:

$$\begin{aligned} v_i(\cdot,0) &= v_{0i}, \quad \nabla v_i(\cdot,0) = \nabla v_{0i}, \quad D^2 v_i(\cdot,0) = D^2 v_{0i}, \\ e_i(\cdot,0) &= e_{0i}, \quad \mathtt{div}\ \mathbb{C}e_i(\cdot,0) = \mathtt{div}\ \mathbb{C}e_{0i}, \quad \mathtt{div}\,\mathtt{div}\,\mathbb{C}e_i(\cdot,0) = \mathtt{div}\,\mathtt{div}\,\mathbb{C}e_{0i}. \end{aligned}$$

We will define the following spaces for the speed of the plates and the strain tensors

$$H := \left\{\begin{array}{ll} \mathbf{v} = (v_1,v_2): & v_i\in \mathtt{H}^2(\omega_i), \quad v_1 = v_2, \quad \text{and} \quad \nabla v_1 = \nabla v_2 \quad \text{on} \quad \gamma_c, \\ & v_i = 0 \quad \text{and} \quad \frac{\partial v_i}{\partial\eta_{\partial\omega_i}} = 0 \quad \text{on} \quad \gamma_{d_i}, \quad i=1,2, \end{array}\right\},$$

$$H^2 := \left\{\begin{array}{ll} \mathbf{e} = (e_1,e_2): & e_i\in \mathtt{H}^2(\omega_i)^{2\times 2}, \quad e_1 = e_2, \quad \text{and} \quad \nabla\,\mathbb{C}e_1 = \nabla\,\mathbb{C}e_2 \quad \text{on} \quad \gamma_c, \\ & (\mathtt{div}\ \mathbb{C}e_i)\cdot\eta_{\partial\omega_i} + \frac{\partial}{\partial\tau_{\partial\omega_i}}((\mathbb{C}e_i)\,\tau_{\partial\omega_i}\cdot\eta_{\partial\omega_i}) = 0 \quad \text{on} \quad \gamma_{n_i}, \\ & (\mathbb{C}e_i)\,\eta_{\partial\omega_i}\cdot\eta_{\partial\omega_i} = 0, \quad \text{on } \gamma_{n_i}, \quad i=1,2 \end{array}\right\}.$$

It should be noted that the values of the functions on the borders γ_{d_i}, γ_c, γ_{n_i}, $i=1,2$ are considered in the sense of the trace.

We will define the internal products and properties that will be used later.

Definition 2.2.1 *For H and H^2 inner products are defined by*

$$\langle \mathbf{v}, \tilde{\mathbf{v}} \rangle_H := \langle v_1, \tilde{v}_1 \rangle_{H^2(\omega_1)} + \langle v_2, \tilde{v}_2 \rangle_{H^2(\omega_2)}, \tag{2.11}$$

$$\langle \mathbf{e} : \tilde{\mathbf{e}} \rangle_{H^2} := \langle e_1 : \tilde{e}_1 \rangle_{H^2(\omega_1)^{2\times 2}} + \langle e_2 : \tilde{e}_2 \rangle_{H^2(\omega_2)^{2\times 2}} = \langle (e_k)_{ij}, (\tilde{e}_k)_{ij} \rangle_{H^2(\omega_k)}, \tag{2.12}$$

respectively. In the last expression the Einstein convention for repeated indices has been used. H and H^2 are closed subspace of $H^2(\omega_1) \times H^2(\omega_2)$ and $H^2(\omega_1)^{2\times 2} \times H^2(\omega_2)^{2\times 2}$, endowed with the inner products in (2.11)-(2.12), they are Hilbert spaces. We will also use the following inner products:

$$\langle \nabla \mathbf{v}; \nabla \tilde{\mathbf{v}} \rangle_{L^2} := \int_{\omega_i} \nabla v_i \cdot \nabla \tilde{v}_i = \langle \nabla v_i; \nabla \tilde{v}_i \rangle_{L^2(\omega_i)}, \tag{2.13}$$

where $\nabla \mathbf{v} = (\nabla v_1, \nabla v_2)$,

$$\langle \mathbf{e} : \tilde{\mathbf{e}} \rangle_{L^{2\times 2}} := \langle e_1 : \tilde{e}_1 \rangle_{L^2(\omega_1)^{2\times 2}} + \langle e_2 : \tilde{e}_2 \rangle_{L^2(\omega_2)^{2\times 2}} = \langle (e_k)_{ij}, (\tilde{e}_k)_{ij} \rangle_{L^2(\omega_k)}. \tag{2.14}$$

Subtly, the punctuation marks ";" and ":" have been introduced to indicate that the inner products should also be made in $\mathbb{R}^2$ and $\mathbb{R}^{2\times 2}$, respectively. For simplicity, the integration variable has been suppressed, however, it is understood that the integral over ω is an area integral and the integral over γ is a line integral.

Lemma 2.2.2 *The H^2 space is a Hilbert space under the inner product defined by*

$$\langle \mathbf{e} : \tilde{\mathbf{e}} \rangle_{H^2_{\mathbb{C}}} := \langle \mathbf{e} : \mathbb{C}\tilde{\mathbf{e}} \rangle_{H^2} = \langle (e_k)_{ij}, (\mathbb{C}_k \tilde{e}_k)_{ij} \rangle_{H^2(\omega_k)}.$$

We will represent the couple $(H^2, \langle \cdot : \cdot \rangle_{H^2_{\mathbb{C}}})$ with $H^2_{\mathbb{C}}$

Proof. $\mathbb{C}$ is a symmetric and positive definite matrix, for this reason, $\langle \cdot : \cdot \rangle_{H^2_{\mathbb{C}}}$ is an inner product in H^2. Furthermore, a Cauchy sequence in $H^2_{\mathbb{C}}$ is a Cauchy sequence in H^2, and this sequence converges in $H^2_{\mathbb{C}}$ if and only if it converges in H^2. ■

It follows the following lemma.

Lemma 2.2.3 *The space $\mathcal{H} := H \times H^2_{\mathbb{C}}$ with the inner product defined by*

$$\langle (\mathbf{v}, \mathbf{e}), (\tilde{\mathbf{v}}, \tilde{\mathbf{e}}) \rangle_{\mathcal{H}} := 12 \langle \mathbf{v}, \tilde{\mathbf{v}} \rangle_H + \langle \mathbf{e} : \tilde{\mathbf{e}} \rangle_{H^2_{\mathbb{C}}} = 12 \langle \mathbf{v}, \tilde{\mathbf{v}} \rangle_H + \langle \mathbf{e} : \mathbb{C}\tilde{\mathbf{e}} \rangle_{H^2}$$

is a Hilbert space.

Definition 2.2.4 *In the coupled model we must define the following plastic flow rule*

$$\begin{gathered}\mathbf{h} \in \partial I_{K(\omega_1)\times K(\omega_2)}(\mathbb{C}_1 e_1, \mathbb{C}_2 e_2) \\ \textit{if and only if} \\ \langle h_i : \mathbb{C}\psi_i - \mathbb{C}_i e_i \rangle_{H^2(\omega_i)} \leq I_{K(\omega_i)}(\mathbb{C}_i \psi_i) - I_{K(\omega_i)}(\mathbb{C}_i e_i)\end{gathered} \tag{2.15}$$

for every $(\mathbb{C}\psi_1, \mathbb{C}\psi_2) \in K(\omega_1) \times K(\omega_2) \subset H^2$.

Notation 2.2.5 *The conventions* $\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e} := (\mathtt{div}\,\mathtt{div}\,\mathbb{C}_1 e_1, \mathtt{div}\,\mathtt{div}\,\mathbb{C}_2 e_2)$, *and* $D^2\mathbf{v} := (D^2 v_1, D^2 v_2)$ *will be used.*

Definition 2.2.6 *On the space* $\mathcal{H}$ *we define the multi-valued operator* $\mathcal{A}$ *by*

$$D(\mathcal{A}) := \big\{(\mathbf{v},\mathbf{e}) \in \mathcal{H} : (\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}\,, D^{\mathbf{2}}\mathbf{v}) \in \mathcal{H}\big\} \tag{2.16}$$

$$(\mathbf{p},\mathbf{s}) \in \mathcal{A}(\mathbf{v},\mathbf{e}) \textit{ if and only if } \begin{cases} \mathbf{p} = -\frac{1}{12}\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e} \\ \mathbf{s} = D^2\mathbf{v} + \mathbf{h} \\ \textit{for some } \mathbf{h} \in \partial I_{K(\omega_1)\times K(\omega_2)}(\mathbb{C}_1 e_1, \mathbb{C}_2 e_2) \end{cases}. \tag{2.17}$$

Also, if $(\mathbf{q},\mathbf{r}) \in \mathcal{H}$ *and* $(\mathbf{v},\mathbf{e}) \in Dom(\mathcal{A})$ *then*

$$\begin{aligned}\langle (\mathbf{q},\mathbf{r}), \mathcal{A}(\mathbf{v},\mathbf{e})\rangle_{\mathcal{H}} &= 12\left\langle \mathbf{q}, -\frac{1}{12}\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}\right\rangle_H + \left\langle \mathbf{r}{:}\mathbf{D^2v} + \mathbf{h}\right\rangle_{H^2_{\mathbb{C}}} \\ &= -\left\langle \mathbf{q}, \mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}\right\rangle_H + \left\langle \mathbb{C}\mathbf{r}{:}\mathbf{D^2v} + \mathbf{h}\right\rangle_{H^2}.\end{aligned}$$

Finally, the problem of transmission of plates with perfect plastic deformation can be defined as a non-linear Cauchy problem.

Problem 2.2.7 *Given* $\mathbf{f}$:$[0,T] \longrightarrow H$ *absolutely continuous, find* $(\mathbf{v}, \mathbb{C}\mathbf{e}) : [0,T] \longrightarrow \mathcal{H}$ *that solve the non-linear Cauchy problem*

$$\begin{gathered}\frac{d}{dt}(\mathbf{v},\mathbf{e}) + \mathcal{A}(\mathbf{v},\mathbf{e}) \ni \mathbf{f} \\ (\mathbf{v},\mathbf{e})(\cdot,0) \in D(\mathcal{A})\end{gathered} \tag{2.18}$$

2.2.2 Existence and Uniqueness of Solutions

Theorem 2.2.8 *The operator* $\mathcal{A}$ *is accretive.*

Proof. First let us see that for all $(\mathbf{v}, \mathbf{e}) \in D(\mathcal{A})$ and $\mathbf{h} \in \partial \mathtt{I}_{K(\omega_1)\times K(\omega_2)}(\mathbb{C}_1(e_1), \mathbb{C}_2(e_2))$, we have

$$-\left\langle \mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, \mathbf{v}\right\rangle_H + \left\langle D^2\mathbf{v} + \mathbf{h} : \mathbf{e}\right\rangle_{H^2_{\mathbb{C}}} = \langle \mathbf{h} : \mathbb{C}\mathbf{e}\rangle_{H^2}. \tag{2.19}$$

In fact, due to Theorem 1.1.3, using convention of summation over repeated indices

$$-\int_{\omega_i} v_i\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}_i e_i + \int_{\omega_i} D^2 v_i : \mathbb{C}_i e_i = -\int_{\partial\omega_i} b_0(\mathbb{C}_i e_i)\, v_i + \int_{\partial\omega_i} b_1(\mathbb{C}_i e_i)\,\frac{\partial v_i}{\partial \eta_{\partial\omega_i}},$$

where
$$\left.\begin{aligned} b_0(\mathbb{C}_i(e_i)) &= \ \mathtt{div}\ \mathbb{C}_i e_i \cdot \eta_{\partial\omega_i} + \tfrac{\partial}{\partial \tau_{\partial\omega_i}}(\mathbb{C}_i e_i)\tau_{\partial\omega_i}\cdot \eta_{\partial\omega_i}) \\ b_1(\mathbb{C}_i(e_i)) &= (\mathbb{C}_i e_i)\eta_{\partial\omega_i}\cdot \eta_{\partial\omega_i}. \end{aligned}\right\}, \ i = 1, 2.$$

On the other side

$$\begin{aligned} -\int_{\partial\omega_i} b_0(\mathbb{C}_i e_i)\, v_i + \int_{\partial\omega_i} b_1(\mathbb{C}_i e_i)\,\frac{\partial v_i}{\partial \eta_{\partial\omega_i}} \quad = \quad & -\int_{\gamma_{d_i}} b_0(\mathbb{C}_i e_i)\, v_i + \int_{\gamma_{d_i}} b_1(\mathbb{C}_i e_i)\,\frac{\partial v_i}{\partial \eta_{\partial\omega_i}} \\ & -\int_{\gamma_{n_i}} b_0(\mathbb{C}_i e_i)\, v_i + \int_{\gamma_{n_i}} b_1(\mathbb{C}_i e_i)\,\frac{\partial v_i}{\partial \eta_{\partial\omega_i}} \\ & -\int_{\gamma_c} b_0(\mathbb{C}_i e_i)\, v_i + \int_{\gamma_c} b_1(\mathbb{C}_i e_i)\,\frac{\partial v_i}{\partial \eta_{\partial\omega_i}}. \end{aligned}$$

From the boundary conditions (2.10), $v_i = 0$ and $\frac{\partial v_i}{\partial \eta_{\partial\omega_i}} = 0$ on γ_{d_i}, we get

$$-\int_{\gamma_{d_i}} b_0(\mathbb{C}_i e_i)\, v_i + \int_{\gamma_{d_i}} b_1(\mathbb{C}_i e_i)\,\frac{\partial v_i}{\partial \eta_{\partial\omega_i}} = 0.$$

On the other hand, on γ_c we have $b_0(\mathbb{C}_1 e_1) = -b_0(\mathbb{C}_2 e_2)$, $b_1(\mathbb{C}_1 e_1) = b_1(\mathbb{C}_2 e_2)$, $v_1 = v_2$, and $\nabla v_1 = \nabla v_2$. Then it follows

$$\int_{\gamma_c} b_0(\mathbb{C}_1 e_1)\, v_1 + \int_{\gamma_c} b_0(\mathbb{C}_2 e_2)\, v_2 = 0, \quad \int_{\gamma_c} b_1(\mathbb{C}_1 e_1)\,\frac{\partial v_1}{\partial \eta_{\partial\omega_1}} + \int_{\gamma_c} b_1(\mathbb{C}_2 e_2)\,\frac{\partial v_2}{\partial \eta_{\partial\omega_2}} = 0.$$

Since $b_0(\mathbb{C}_i e_i) = 0$, $b_1(\mathbb{C}_i e_i) = 0$ on γ_{n_i}, we obtain

$$\left\langle -\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, \mathbf{v}\right\rangle + \left\langle D^2\mathbf{v}, \mathbb{C}\mathbf{e}\right\rangle = -\int_{\omega_i} v_i\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}_i e_i + \int_{\omega_i} D^2 v_i : \mathbb{C}_i e_i = 0. \tag{2.20}$$

What we want to prove is

$$\left\langle -\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, \mathbf{v}\right\rangle_H + \left\langle D^2\mathbf{v} : \mathbf{e}\right\rangle_{H^2_{\mathbb{C}}} = 0, \tag{2.21}$$

and for this we must prove analogous identities to (2.20) for the derivatives of $\mathbf{v}$ and $\mathbf{e}$. That is, we must prove

$$\langle -D\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, D\mathbf{v}\rangle + \left\langle\,\mathtt{div}\,D^2\mathbf{v},\,\mathtt{div}\,\mathbb{C}\mathbf{e}\right\rangle = 0, \tag{2.22}$$

$$\left\langle -D^2\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, D^2\mathbf{v}\right\rangle + \left\langle\mathtt{div}\,\mathtt{div}\,D^2\mathbf{v},\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}\right\rangle = 0. \tag{2.23}$$

From the divergence property

$$\mathtt{div}\,\left(\mathtt{div}\,\mathtt{div}\,\mathbb{C}e\,.\,Dv\right) = D\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}e\cdot Dv + \mathtt{div}\,\mathtt{div}\,Ce\,.\,\Delta v,$$

and the divergence theorem, we get

$$\begin{aligned}\left\langle D\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}e^i, Dv^i\right\rangle &+ \left\langle\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}e^i, \Delta v^i\right\rangle\\ &= \int_{\omega_i}\mathtt{div}\,\left(\mathtt{div}\,\mathtt{div}\,\mathbb{C}e^i\,.\,Dv^i\right)\\ &= \int_{\partial\omega_i}\left(\mathtt{div}\,\mathtt{div}\,\mathbb{C}e^i\,.\,Dv^i\right)\cdot\eta_{\partial\omega_i}\\ &= \left\{\int_{\gamma_c}+\int_{\gamma_{d_i}}+\int_{\gamma_{n_i}}\right\}\left(\mathtt{div}\,\mathtt{div}\,\mathbb{C}e^i\,.\,Dv^i\right)\cdot\eta_{\partial\omega_i}.\end{aligned}$$

From (2.16), $\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}$, $D^2\mathbf{v}$ also verify (2.10), for such,

$$\left\{\int_{\gamma_c}+\int_{\gamma_{d_i}}+\int_{\gamma_{n_i}}\right\}\left(\mathtt{div}\,\mathtt{div}\,\mathbb{C}e^i\,.\,Dv^i\right)\cdot\eta_{\partial\omega_i} = \int_{\gamma_c}\left(\mathtt{div}\,\mathtt{div}\,\mathbb{C}e^i\,.\,Dv^i\right)\cdot\eta_{\partial\omega_i}.$$

Since $\eta_{\partial\omega_1} = -\eta_{\partial\omega_2}$, by (2.9) we obtain

$$\int_{\gamma_c}\left(\mathtt{div}\,\mathtt{div}\,\mathbb{C}e^i\,.\,Dv^i\right)\cdot\eta_{\partial\omega_i} = 0,$$

and

$$\langle D\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, D\mathbf{v}\rangle = -\left\langle\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, \Delta\mathbf{v}\right\rangle. \tag{2.24}$$

We can commute $\mathtt{div}\,(D^2v^i) = D(\,\mathtt{div}\,Dv^i)$, by such,

$$\left\langle\mathtt{div}\,\mathbb{C}\mathbf{e},\,\mathtt{div}\,(D^2\mathbf{v})\right\rangle = \langle\mathtt{div}\,\mathbb{C}\mathbf{e}, D(\,\mathtt{div}\,D\mathbf{v})\rangle = \langle\mathtt{div}\,\mathbb{C}\mathbf{e}, D\Delta\mathbf{v}\rangle\,. \tag{2.25}$$

Since $\langle \mathtt{div}\,\mathbb{C}e^i, D\Delta v^i\rangle = -\langle \mathtt{div}\,\mathtt{div}\,\mathbb{C}e^i, \Delta v^i\rangle + \int_{\partial\omega}(\mathtt{div}\,\mathbb{C}e^i \,\boldsymbol{.}\, \Delta v^i)\cdot\eta_{\partial\omega_i}$, by (2.9)-(2.10),

$$\langle \mathtt{div}\,\mathbb{C}\mathbf{e}, D\Delta\mathbf{v}\rangle = -\langle \mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, \Delta\mathbf{v}\rangle\,. \tag{2.26}$$

Equating (2.25) and (2.26),

$$\left\langle \mathtt{div}\,\mathbb{C}\mathbf{e},\, \mathtt{div}\,(D^2\mathbf{v})\right\rangle = -\langle \mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, \Delta\mathbf{v}\rangle\,. \tag{2.27}$$

Finally, for (2.24), (2.27), we get (2.22).

Permuting D and $\mathtt{div}$ in $\mathtt{div}\,\mathtt{div}\,(D^2 v^i)$, we have

$$\begin{aligned}\left\langle \mathtt{div}\,\mathtt{div}\,\mathbb{C}e^i,\, \mathtt{div}\,\mathtt{div}\,(D^2 v^i)\right\rangle &= \left\langle \mathtt{div}\,\mathtt{div}\,\mathbb{C}e^i,\, \mathtt{div}\,D\,\mathtt{div}\,(Dv^i)\right\rangle \\ &= \left\langle \mathtt{div}\,\mathtt{div}\,\mathbb{C}e^i, \Delta^2 v^i\right\rangle.\end{aligned} \tag{2.28}$$

For $w \in C^1(\omega;\mathbb{R}^2)$ and $W \in C^1(\omega;\mathbb{R}^{2\times 2})$ it can be verified that

$$\mathtt{div}\,(wW) = Dw : W + w\cdot\,\mathtt{div}\,W,$$

and by the divergence theorem,

$$\int_\omega Dw : W + w\cdot\,\mathtt{div}\,W = \int_\omega \mathtt{div}\,(wW) = \int_{\partial\omega} wW\cdot n.$$

Considering $w = D\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}e^i$, $W = D^2 v^i$, and the boundary conditions (2.9)-(2.10), we get $\sum\int_{\partial\omega}(D\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}e^i)\,(D^2 v^i)\cdot\eta_{\partial\omega_i} = 0$, and

$$\begin{aligned}\left\langle D^2\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, D^2\mathbf{v}\right\rangle &= -\left\langle D\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, \mathtt{div}D^2\mathbf{v}\right\rangle \\ &= -\langle D\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, D\Delta\mathbf{v}\rangle\,.\end{aligned} \tag{2.29}$$

Again by integration by parts and the boundary conditions (2.9)-(2.10) on the term (2.29),

$$\left\langle D^2\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, D^2\mathbf{v}\right\rangle = \left\langle \mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, \Delta^2\mathbf{v}\right\rangle. \tag{2.30}$$

From the equations 2.28 and 2.30 we have

$$\left\langle \mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e},\, \mathtt{div}\,\mathtt{div}\,(D^2\mathbf{v})\right\rangle + \left\langle -D^2\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, D^2\mathbf{v}\right\rangle = 0.$$

Finally, by (2.20), (2.22) and (2.23), we obtain (2.21), which proves

$$\langle (\mathbf{v},\mathbf{e}), \mathcal{A}(\mathbf{v},\mathbf{e})\rangle_{\mathcal{H}} = -\left\langle \mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, \mathbf{v}\right\rangle_H + \left\langle D^2\mathbf{v} + \mathbf{h} : \mathbf{e}\right\rangle_{H^2_{\mathbb{C}}} = \langle \mathbf{h} : \mathbb{C}\mathbf{e}\rangle_{H^2}.$$

To prove the accretivity of $\mathcal{A}$ consider $(\mathbf{v},\mathbf{e}), (\hat{\mathbf{v}},\hat{\mathbf{e}}) \in Dom(\mathcal{A})$. Then by the linearity of $\mathtt{div}\,\mathtt{div}$ and D^2

$$-\left\langle \mathtt{div}\,\mathtt{div}\,\mathbb{C}(\mathbf{e}-\hat{\mathbf{e}}), (\mathbf{v}-\hat{\mathbf{v}})\right\rangle_H + \left\langle D^2(\mathbf{v}-\hat{\mathbf{v}}) + (\mathbf{h}-\hat{\mathbf{h}}) : (\mathbf{e}-\hat{\mathbf{e}})\right\rangle_{H^2_{\mathbb{C}}} = \left\langle \mathbf{h}-\hat{\mathbf{h}} : \mathbb{C}\mathbf{e} - \mathbb{C}\hat{\mathbf{e}}\right\rangle_{H^2},$$

where $\mathbf{h} \in \partial \mathtt{I}_{K(\omega_1)\times K(\omega_2)}(\mathbb{C}_1 e_1, \mathbb{C}_2 e_2)$ and $\hat{\mathbf{h}} \in \partial \mathtt{I}_{K(\omega_1)\times K(\omega_2)}(\mathbb{C}_1 \hat{e}_1, \mathbb{C}_2 \hat{e}_2)$. Then

$$\langle (\mathbf{v},\mathbf{e}) - (\hat{\mathbf{v}},\hat{\mathbf{e}}), \mathcal{A}(\mathbf{v},\mathbf{e}) - \mathcal{A}(\hat{\mathbf{v}},\hat{\mathbf{e}})\rangle_{\mathcal{H}} = \left\langle \mathbf{h}-\hat{\mathbf{h}} : \mathbb{C}\mathbf{e} - \mathbb{C}\hat{\mathbf{e}}\right\rangle_{H^2},$$

and by Theorem 1.1.12, $\partial \mathtt{I}_{K(\omega_1)\times K(\omega_2)}$ is accretive, which implies that $\mathcal{A}$ is accretive. ■

Theorem 2.2.9 *The operator $\mathcal{A}$ is m-accretive.*

Proof. $\mathcal{A}$ is accretive by Theorem 2.2.8, we will prove that $Rg(\mathtt{I} + \mathcal{A}) = \mathcal{H}$; that is, given $(\mathbf{q},\mathbf{r}) \in \mathcal{H}$, there is $(\mathbf{v}_0,\mathbf{e}_0) \in D(\mathcal{A})$ such that

$$(\mathtt{I} + \mathcal{A})(\mathbf{v}_0,\mathbf{e}_0) \ni (\mathbf{q},\mathbf{r}). \tag{2.31}$$

Let $(\mathbf{q},\mathbf{r})$ be an arbitrary element of $\mathcal{H}$. Consider the functional

$$F(\mathbf{e}) = \frac{1}{2}\langle \mathbf{e}, \mathbb{C}\mathbf{e}\rangle + \frac{1}{24}\left\langle \mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}, \mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}\right\rangle + \langle \mathbf{q}, \mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}\rangle + \mathtt{I}_K(\mathbb{C}\mathbf{e}) - \langle \mathbf{r}, \mathbb{C}\mathbf{e}\rangle.$$

It is easy to see that F is convex and lower semicontinuous on $H^2_{\mathbb{C}}$. Also,

$$\lim_{\|\mathbf{e}\|_{H^2_{\mathbb{C}}}\to\infty} F(\mathbf{e}) = +\infty.$$

By Theorem 1.1.9 and the hypotheses about $K := K(\omega_1) \times K(\omega_2)$ (K closed and convex), F has a minimum at $\mathbf{e}_0 \in K$, and therefore $\delta F(\mathbf{e}_0, \tilde{\mathbf{e}})_{|\varepsilon=0} = 0$ for all $\tilde{\mathbf{e}} \in H^2_{\mathbb{C}}$ (see [29, Ch 1]), where

$$\delta F(\mathbf{e}_0,\tilde{\mathbf{e}}) := \frac{d}{d\varepsilon} F(\mathbf{e}_0 + \varepsilon\tilde{\mathbf{e}})\Big|_{\varepsilon=0}$$

is the first variation of F. In that case, if $\mathbf{h} \in \partial\, \mathtt{I}_{K(\omega_1)\times K(\omega_2)}(\mathbb{C}\mathbf{e}_0)$ and

$$\mathbf{v}_0 := \tfrac{1}{12}\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}_0 + \mathbf{q}, \tag{2.32}$$

then

$$\frac{d}{d\varepsilon}F(\mathbf{e}_0+\varepsilon\tilde{\mathbf{e}}) = \langle \mathbf{e}_0, \mathbb{C}\tilde{\mathbf{e}}\rangle + \left\langle \tfrac{1}{12}\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}_0,\ \mathtt{div}\,\mathtt{div}\,\mathbb{C}\tilde{\mathbf{e}}\right\rangle + \langle \mathbf{q},\ \mathtt{div}\,\mathtt{div}\,\mathbb{C}\tilde{\mathbf{e}}\rangle + \langle \mathbf{h}, \mathbb{C}\tilde{\mathbf{e}}\rangle - \langle \mathbf{r}, \mathbb{C}\tilde{\mathbf{e}}\rangle$$

$$\begin{aligned}
&= \langle \mathbf{e}_0, \mathbb{C}\tilde{\mathbf{e}}\rangle + \left\langle \tfrac{1}{12}\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}_0 + \mathbf{q},\ \mathtt{div}\,\mathtt{div}\,\mathbb{C}\tilde{\mathbf{e}}\right\rangle + \langle \mathbf{h}, \mathbb{C}\tilde{\mathbf{e}}\rangle - \langle \mathbf{r}, \mathbb{C}\tilde{\mathbf{e}}\rangle \\
&= \langle \mathbf{e}_0, \mathbb{C}\tilde{\mathbf{e}}\rangle + \langle \mathbf{v}_0,\ \mathtt{div}\,\mathtt{div}\,\mathbb{C}\tilde{\mathbf{e}}\rangle + \langle \mathbf{h}, \mathbb{C}\tilde{\mathbf{e}}\rangle - \langle \mathbf{r}, \mathbb{C}\tilde{\mathbf{e}}\rangle \\
&= \langle \mathbf{e}_0, \mathbb{C}\tilde{\mathbf{e}}\rangle + \left\langle D^2\mathbf{v}_0, \mathbb{C}\tilde{\mathbf{e}}\right\rangle + \langle \mathbf{h}, \mathbb{C}\tilde{\mathbf{e}}\rangle - \langle \mathbf{r}, \mathbb{C}\tilde{\mathbf{e}}\rangle = 0, \quad \text{for all } \tilde{\mathbf{e}} \in H^2_{\mathbb{C}},
\end{aligned}$$

and

$$\mathbf{r} = \mathbf{e}_0 + D^2\mathbf{v}_0 + \mathbf{h}, \quad \mathbf{h} \in \partial\, \mathtt{I}_{K(\omega_1)\times K(\omega_2)}(\mathbb{C}\mathbf{e}_0). \tag{2.33}$$

By (2.32) and (2.33),

$$(I+\mathcal{A})(\mathcal{H}) \supseteq (\mathbf{v}_0, \mathbf{e}_0) + \left(-\tfrac{1}{12}\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}\,, D^2\mathbf{v} + \partial\, \mathtt{I}_K(\mathbb{C}\mathbf{e}_i)\right) \ni (\mathbf{q}, \mathbf{r}).$$

Using this last relation together with the Theorem 2.2.8, we get

$$\mathcal{A} : (\mathbf{v}, \mathbf{e}) \longmapsto \left(-\tfrac{1}{12}\,\mathtt{div}\,\mathtt{div}\,\mathbb{C}\mathbf{e}\,, D^2\mathbf{v} + \partial\, \mathtt{I}_K(\mathbb{C}\mathbf{e}_i)\right)$$

is m-accretive. ■

Corollary 2.2.10 *Given* $\mathbf{f} \in W^{1,1}(0,T;\mathcal{H})$, $(\mathbf{v},\mathbf{e})(0,\cdot) \in D(\mathcal{A})$, *there is a single function* $(\mathbf{v},\mathbf{e}) \in W^{1,\infty}(0,T;\mathcal{H})$ *that solve the non-linear Cauchy problem*

$$\frac{d}{dt}(\mathbf{v},\mathbf{e}) + \mathcal{A}(\mathbf{v},\mathbf{e}) \ni \mathbf{f} \tag{2.34}$$

$$(\mathbf{v},\mathbf{e})(0,\cdot) \in D(\mathcal{A}).$$

Proof. Since A is m-accretive, the result follows from Theorem 1.1.15. ■

Chapter 3

Adhesive Contact Problem with Elastoplasticity

Quasi-static problems of adhesive contact with elastoplasticity have focused on energetic solutions, for example, the case of contact and elastoplasticity with hardening [50], delamination problems [33, 49], the rate-independent model with damage [7], adhesive contact with temperature [47], the numerical approach developed in [44], among others. This chapter considers the elements developed in Section 1.2.1 on linear elastoplasticity in the rate-independent system, and the Section 1.2.2 about adhesive contact. After defining a space of solutions according to the boundary conditions of the adhesive contact, we will formulate the problem weakly from the differential inclusions that define the elastoplastic deformation and the unilateral and unidirectional contact. Indirectly we will prove the existence of weak solutions using Colli [13] and its results on doubly nonlinear problems. The results of this Chapter were published in Peñas [46].

3.1 Abstract Formulation of the Model

Two deformable solids occupying reference domains Ω_1, Ω_2 are considered in adhesive contact in a common region $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ (see Figure 3.1). Both solids with a boundary $\Gamma_{0i} \subset \partial\Omega_i \backslash \Gamma$ with prescribed displacement ($u^i_{|\Gamma_{0i}} = 0$, $i = 1, 2$). For each material, we consider a dissipation potential $\mathtt{R} : U \to]-\infty, +\infty]$ and a stored energy density $\mathtt{W} : U \to]-\infty, +\infty[$, where U is a Hilbert space for (e, p)[1]. The elastoplastic

[1] e is the linearized symmetric strain tensor and p is the plastic tensor.

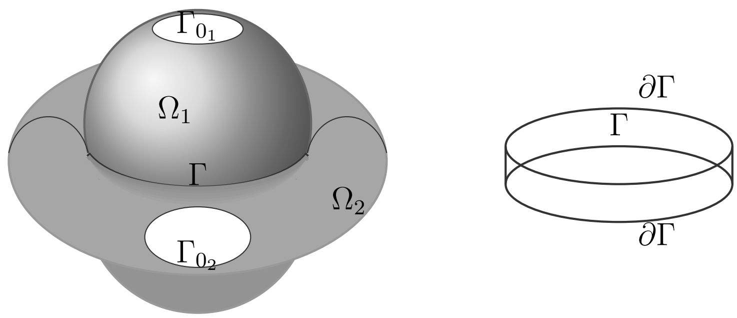

Figure 3.1: Ω_1 adhered to Ω_2 in Γ, with prescribed displacement in $\Gamma_{0_1} \cup \Gamma_{0_2}$.

component is defined by a momentum equation and a plastic flow rule as a function of $\mathtt{W}$ and $\mathtt{R}$. The treatment in this work does not include damage and temperature. We consider unidirectional (irreversible) and unilateral (no penetration between solids) adhesive contact through differential Fremond inclusions on the contact boundary (see e.g. [25]). The bonding field β, the displacements on the boundary $u_{|\Gamma}$ and indicator functions define these differential inclusions.

3.2 Weak Formulation

We define the space W as the set of triples

$$(\mathtt{u}, \mathtt{p}, \beta)_W := \begin{pmatrix} \mathtt{u} \\ \mathtt{p} \\ \beta \end{pmatrix}$$

such that $\mathtt{u} = (u^1, u^2) \in L^2(\Omega_1) \times L^2(\Omega_2)$, $\mathtt{p} = (p^1, p^2) \in L^2(\Omega_1)^{3\times3} \times L^2(\Omega_2)^{3\times3}$, $\beta \in L^2(\Gamma)$. Since Ω_1 and Ω_2 have boundaries of class C^1, the elements of $H^1(\Omega_1)^3 \times H^1(\Omega_2)^3$ can be extended to the boundary from the trace operator γ_0 (see e.g. [22]). The representation of this extension will be the pair $(\mathtt{u}_{|\partial\Omega\setminus\Gamma}, \mathtt{u}_{|\Gamma})$, where $\mathtt{u}_{|\partial\Omega\setminus\Gamma} := (\gamma_0 \mathtt{u})_{|\partial\Omega\setminus\Gamma}$, $\mathtt{u}_{|\Gamma} := (\gamma_0 \mathtt{u})_{|\Gamma}$. We define

$$H^1 := \left\{ \mathtt{u} \in H^1(\Omega_1)^3 \times H^1(\Omega_2)^3 : \nabla u^i = \left(\nabla u^i\right)^\intercal, \, i = 1, 2 \right\},$$

$$H^{1/2} \ : \ = \left\{ \begin{array}{r} (\mathtt{u}_{|\partial\Omega\setminus\Gamma}, \mathtt{u}_{|\Gamma}) : \quad \mathtt{u}_{|\Gamma} = \mathtt{u}_{|\partial\Omega\setminus\Gamma} \text{ in } \partial\Gamma, \, u^i_{|\partial\Omega\setminus\Gamma} \in L^2(\partial\Omega_i\setminus\Gamma)^3, \\ u^i_{|\Gamma} \in L^2(\Gamma)^3, \, i = 1, 2 \end{array} \right\},$$

and we represent with V the array space of vector functions $(\mathtt{u},\mathtt{p},\beta)_V$, where

$$(\mathtt{u},\mathtt{p},\beta)_V := \begin{pmatrix} \mathtt{u} & \mathtt{e(u)} & \mathtt{u}_{|\partial\Omega\setminus\Gamma} & \mathtt{u}_{|\Gamma} \\ & & \mathtt{p} & \\ & \beta & \nabla\beta & \end{pmatrix},$$

$\mathtt{u} \in H^1 \times H^{1/2}$, $\mathtt{p} \in L^2(\Omega_1)^{3\times3}_{dev} \times L^2(\Omega_2)^{3\times3}_{dev}$, $\beta \in H^1(\Gamma)$. For short we will use the notation $\langle\cdot,\cdot\rangle$ to indicate any of the products

$$\langle\cdot,\cdot\rangle_{L^2(\Omega_1)^3\times L^2(\Omega_2)^3} \qquad \text{or} \qquad \langle\cdot,\cdot\rangle_{L^2(\Omega_1)^{3\times3}\times L^2(\Omega_2)^{3\times3}},$$

and $\mathtt{u}_{|\Gamma}\binom{-1}{1}$ to represent $u^2_\Gamma - \hat{u}^1_\Gamma$.

Theorem 3.2.1 *V and W are Hilbert space under the inner product*

$$\Big\langle (\mathtt{u},\mathtt{p},\beta)_V, (\hat{\mathtt{u}},\hat{\mathtt{p}},\hat{\beta})_V \Big\rangle_V := \langle \mathtt{u},\hat{\mathtt{u}}\rangle_{H^1} + \langle \gamma_0\mathtt{u},\gamma_0\hat{\mathtt{u}}\rangle_{H^{1/2}} + \langle \mathtt{p},\hat{\mathtt{p}}\rangle + \Big\langle \beta,\hat{\beta}\Big\rangle_{H^1(\Gamma)},$$

$$\Big\langle (\mathtt{u},\mathtt{p},\beta)_W, (\hat{\mathtt{u}},\hat{\mathtt{p}},\hat{\beta})_W \Big\rangle_W := \langle \mathtt{u},\hat{\mathtt{u}}\rangle + \langle \mathtt{p},\hat{\mathtt{p}}\rangle + \Big\langle \beta,\hat{\beta}\Big\rangle_{L^2(\Gamma)}.$$

Besides that, $V \Subset W$.

Theorem 3.2.2 *If* $\mathtt{u}$, $\mathtt{p}$, β *satisfy* (1.6),(1.7), (1.9), (1.10)-(1.13), $\mathtt{q} = (q^1, q^2) \in \left(\partial\,\mathtt{R}(\dot{p}^1), \partial\,\mathtt{R}(\dot{p}^2)\right)$, $\varrho \in \partial\,\mathtt{I}_{\text{-}}(\mathtt{u}_{|\Gamma}\binom{-1}{1}\cdot \mathtt{N}_2)$, $\alpha \in \partial\,\mathtt{I}_{\text{-}}(\dot{\beta})$, *and* $\varsigma \in \partial\,\mathtt{I}_{[0,1]}(\beta)$, *then for all* $(\hat{\mathtt{u}},\hat{\mathtt{p}},\hat{\beta})_V \in V$,

$$\Big\langle \partial_e \mathtt{W}(\mathtt{e(u)},\mathtt{p})\,,\,\mathtt{e}(\hat{\mathtt{u}}) \Big\rangle - \int_\Gamma \sigma^i\,\mathtt{N}^i\cdot\hat{u}^i ds = \Big\langle \mathtt{f}\,,\,\hat{\mathtt{u}} \Big\rangle_{H^1} + \Big\langle \mathtt{g}\,,\,\hat{\mathtt{u}}_{|\partial\Omega\setminus\Gamma} \Big\rangle_{\partial\Omega\setminus\Gamma} \tag{3.1}$$

$$\Big\langle \mathtt{q}\,,\,\hat{\mathtt{p}} \Big\rangle + \Big\langle \partial_p\mathtt{W}(\mathtt{e(u)},\mathtt{p})\,,\,\hat{\mathtt{p}} \Big\rangle = 0 \tag{3.2}$$

$$\int_\Gamma \sigma^i\,\mathtt{N}_i\cdot\hat{u}^i ds + k\Big\langle\, \beta\,\mathtt{u}_{|\Gamma}\tbinom{-1}{1}\,,\,\hat{\mathtt{u}}_{|\Gamma}\tbinom{-1}{1}\,\Big\rangle_\Gamma + \int_\Gamma \varrho\,\hat{\mathtt{u}}_{|\Gamma}\tbinom{-1}{1}\cdot\mathtt{N}_2\,ds = 0 \tag{3.3}$$

$$\Big\langle c_s\dot{\beta} + \alpha + \varsigma, \hat{\beta}\Big\rangle_\Gamma + k_s\Big\langle \nabla\beta, \nabla\hat{\beta}\Big\rangle_\Gamma + \tfrac{k}{2}\Big\langle\, \|\,\mathtt{u}_{|\Gamma}\tbinom{-1}{1}\,\|^2, \hat{\beta}\,\Big\rangle_\Gamma = \Big\langle w_s, \hat{\beta}\Big\rangle_\Gamma. \tag{3.4}$$

Proof. By Green's formula,

$$\Big\langle\ \sigma^i,\mathtt{e}(\hat u_i)\ \Big\rangle + \Big\langle\ \mathtt{div}\,\sigma^i,\hat u^i\ \Big\rangle = \int_{\partial\Omega_i} \sigma^i\,\mathtt{N}_i\cdot\hat u^i ds, \quad i=1,2. \tag{3.5}$$

Substituting (1.6) into (3.5), where $\sigma^i = \partial_e \mathtt{W}(\mathtt{e}(u^i),p^i)$, we obtain

$$\begin{aligned}\int_{\Omega_i} f^i\cdot\hat u^i &= -\Big\langle\ \mathtt{div}\,\sigma^i\,,\hat u^i\ \Big\rangle\\ &= \Big\langle\ \partial_e\mathtt{W}(\mathtt{e}(u^i),p^i)\,,\,\mathtt{e}(\hat u^i)\ \Big\rangle - \int_{\partial\Omega_i}\sigma^i\mathtt{N}_i\cdot\hat u^i ds, \quad i=1,2.\end{aligned} \tag{3.6}$$

From (1.9),

$$\begin{aligned}\int_{\partial\Omega_i}\sigma^i\mathtt{N}_i\cdot\hat u^i ds &= \int_\Gamma \sigma^i\mathtt{N}_i\cdot\hat u^i ds + \int_{\partial\Omega_i\setminus\Gamma}\sigma^i\mathtt{N}_i\cdot\hat u^i ds\\ &= \int_\Gamma \sigma^i\mathtt{N}_i\cdot\hat u^i ds + \int_{\partial\Omega_i\setminus(\Gamma\cup\Gamma_i)} g^i\cdot\hat u^i,\end{aligned} \tag{3.7}$$

and from the substitution of (3.7) in (3.6),

$$\Big\langle\ \partial_e\mathtt{W}(\mathtt{e}(\mathtt{u}),\mathtt{p})\,,\,\mathtt{e}(\hat{\mathtt{u}})\ \Big\rangle - \int_\Gamma \sigma^i\mathtt{N}_i\cdot\hat u^i ds = \Big\langle\ \mathtt{f}\,,\,\hat{\mathtt{u}}\ \Big\rangle + \Big\langle\ \mathtt{g}\,,\,\hat{\mathtt{u}}\ \Big\rangle_{\partial\Omega\setminus\Gamma}.$$

Equation (3.2) is immediate by (1.7). Multiplying the first equation of (1.11) by $\hat u^1$, the second by $\hat u^2$, and integrating over Γ,

$$\int_\Gamma \Big\{\ (\sigma^1\,\mathtt{N}_1\cdot\hat u^1 + \sigma^2\,\mathtt{N}_2\cdot\hat u^2) + k\beta\,\left(u^2-u^1\right)\cdot\left(\hat u^2-\hat u^1\right) + \varrho\,\left(\hat u^2-\hat u^1\right)\cdot\mathtt{N}_2\ \Big\}\,ds$$

$$= \int_\Gamma \sigma^i\,\mathtt{N}_i\cdot\hat u^i ds + k\Big\langle\ \beta\,\mathtt{u}_{|\Gamma}\left({}^{-1}_{\ 1}\right),\,\hat{\mathtt{u}}_{|\Gamma}\left({}^{-1}_{\ 1}\right)\ \Big\rangle_\Gamma + \int_\Gamma \varrho\,\hat{\mathtt{u}}_{|\Gamma}\left({}^{-1}_{\ 1}\right)\cdot\mathtt{N}_2\,ds = 0,$$

where $\varrho \in\partial\,\mathtt{I}_{\text{-}}(\mathtt{u}_{|\Gamma}\left({}^{-1}_{\ 1}\right)\,\cdot\mathtt{N}_2)$, $\hat{\mathtt{u}} = (\hat u^1,\hat u^2)\in H^1$.

If $\beta\in H^1(\Gamma)$, $\mathtt{u}\in H^1$, $c_s\dot\beta+\alpha\in\partial\left(\frac{c_s}{2}\dot\beta^{\,2} + \mathtt{I}_{\text{-}}(\dot\beta)\right)$ and $\varsigma\in\partial\,\mathtt{I}_{[0,1]}(\beta)$ satisfies (1.10), then for all $\hat\beta\in H^1_\Gamma$

$$\Big\langle c_s\dot\beta+\alpha,\hat\beta\Big\rangle_\Gamma - k_s\Big\langle\Delta\beta,\hat\beta\Big\rangle_\Gamma + \Big\langle\varsigma,\hat\beta\Big\rangle_\Gamma = \Big\langle w_s,\hat\beta\Big\rangle_\Gamma - \tfrac12 k\Big\langle |\,u^2-u^1\,|^2,\hat\beta\Big\rangle_\Gamma.$$

Now, by (1.12), we have

$$\left\langle c_s\dot{\beta}+\alpha,\hat{\beta}\right\rangle_\Gamma + k_s\left\langle\nabla\beta,\nabla\hat{\beta}\right\rangle_\Gamma + \left\langle\varsigma,\hat{\beta}\right\rangle_\Gamma = \left\langle w_s,\hat{\beta}\right\rangle_\Gamma - \tfrac{k}{2}\left\langle \|\mathtt{u}_{|\Gamma}\binom{-1}{1}\|^2,\hat{\beta}\right\rangle_\Gamma . \quad \blacksquare$$

We will write the equations of the Theorem 3.2.2 so that the adhesive contact model can be represented as a doubly non-linear problem. The sum of the terms (3.1) - (3.4) will be grouped taking into account the pairings (V, V^*), and (W, W^*).

$$\left\{\left\langle 0\,,\,\hat{\mathtt{u}}\right\rangle + \left\langle \mathtt{q}\,,\,\hat{\mathtt{p}}\right\rangle + \left\langle c_s\dot{\beta}+\alpha,\hat{\beta}\right\rangle_\Gamma\right\}$$

$$+\left\{\begin{array}{c} \left\langle 0\,,\,\hat{\mathtt{u}}\right\rangle + \left\langle \partial_e\mathtt{W}(\mathtt{e}(\mathtt{u}),\mathtt{p})\,,\,\mathtt{e}(\hat{\mathtt{u}})\right\rangle + \left\langle 0\,,\,\hat{\mathtt{u}}_{|\partial\Omega\setminus\Gamma}\right\rangle_{\partial\Omega\setminus\Gamma} \\ +k\left\langle \beta\,\mathtt{u}_{|\Gamma}\binom{-1}{1},\,\hat{\mathtt{u}}_{|\Gamma}\binom{-1}{1}\right\rangle_\Gamma + \int_\Gamma \varrho\,\hat{\mathtt{u}}_{|\Gamma}\binom{-1}{1}\cdot\mathtt{N}_2\,ds \\ +\left\langle \partial_p\mathtt{W}(\mathtt{e}(\mathtt{u}),\mathtt{p})\,,\,\hat{\mathtt{p}}\right\rangle + \left\langle \varsigma + \|\mathtt{u}_{|\Gamma}\binom{-1}{1}\|^2,\hat{\beta}\right\rangle_\Gamma + k_s\left\langle\nabla\beta,\nabla\hat{\beta}\right\rangle_\Gamma \end{array}\right\}$$

$$=\left\{\begin{array}{c} \left\langle \mathtt{f}\,,\,\hat{\mathtt{u}}\right\rangle + \left\langle 0\,,\,\mathtt{e}(\hat{\mathtt{u}})\right\rangle \\ +\left\langle \mathtt{g}\,,\,\hat{\mathtt{u}}_{|\partial\Omega\setminus\Gamma}\right\rangle_{\partial\Omega\setminus\Gamma} + \left\langle 0\,,\,\hat{\mathtt{u}}_{|\Gamma}\right\rangle_\Gamma \\ +\left\langle 0\,,\,\hat{\mathtt{p}}\right\rangle + \left\langle \omega_s,\hat{\beta}\right\rangle_\Gamma + \left\langle 0,\nabla\hat{\beta}\right\rangle_\Gamma \end{array}\right\},$$

where $\mathtt{f} = (f^1, f^2)$, $\mathtt{g} = (g^1, g^2)$. The first summation corresponds to the product $\left\langle (0,\mathtt{q}, c_s\dot{\beta}+\alpha)_W\ ,\ (\hat{\mathtt{u}},\hat{\mathtt{p}},\hat{\beta})_W\right\rangle_W$, where

$$(0,\mathtt{q},c_s\dot{\beta}+\alpha)_W \in \partial_W\left\{\mathtt{R}(\dot{\mathtt{p}}) + \frac{c_s}{2}\left(\dot{\beta}\right)^2 + \mathtt{I}_-(\dot{\beta})\right\}, \tag{3.8}$$

while the second adding corresponds to the product $\left\langle v,\ (\hat{\mathtt{u}},\hat{\mathtt{p}},\hat{\beta})_V\right\rangle_V$, where

$$v \in \partial_V\left\{\mathtt{W}(\mathtt{e},\mathtt{p}) + \tfrac{k_s}{2}\|\nabla\beta\|^2 + \tfrac{k}{2}\beta\|\mathtt{u}_{|\Gamma}\binom{-1}{1}\|^2 + \mathtt{I}_{[0,1]}(\beta) + \mathtt{I}_-(\mathtt{u}_{|\Gamma}\binom{-1}{1}\cdot\mathtt{N}_2)\right\}. \tag{3.9}$$

The last term is the product between F y $(\hat{\mathtt{u}},\hat{\mathtt{p}},\hat{\beta})_V$, where

$$F := \begin{pmatrix} \mathtt{f} & 0 & \mathtt{g} & 0 \\ & & 0 & \\ & \omega_s & 0 & \end{pmatrix}.$$

The above calculations motivate the following definitions.

Definition 3.2.3 *We define by $\varphi : W \to]-\infty,\infty]$, and $\psi : V \to]-\infty,\infty]$ the functional*

$$\varphi(\mathtt{u},\mathtt{p},\beta)_W := \mathtt{R}(\mathtt{p}) + \tfrac{c_s}{2}\,(\beta)^2 + \mathtt{I}_{\text{-}}(\beta),$$

$$\psi(\mathtt{u},\mathtt{p},\beta)_V := \mathtt{W}(\mathtt{e},\mathtt{p}) + \tfrac{k_s}{2}\,\|\,\nabla\beta\,\|^2 + \tfrac{k}{2}\beta\,\|\,\mathtt{u}_{|\Gamma}\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)\|^2 + \mathtt{I}_{[0,1]}(\beta) + \mathtt{I}_{\text{-}}(\mathtt{u}_{|\Gamma}\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)\cdot\mathtt{N}_2),$$

and their respective subdifferentials $\partial_W\varphi : W \rightrightarrows W^*$, and $\partial_V\psi : V \rightrightarrows V^*$ by

$$\partial_W\varphi(\mathtt{u},\mathtt{p},\beta)_W := \begin{pmatrix} 0 \\ \partial_{\mathtt{p}}\mathtt{R}(\mathtt{p}) \\ c_s\beta + \partial_\beta\mathtt{I}_{\text{-}}(\beta)\end{pmatrix},$$

$$\partial_V\psi(\mathtt{u},\mathtt{p},\beta)_V$$

$$:= \begin{pmatrix} 0 \quad \partial_{\mathtt{e}}\mathtt{W}(\mathtt{e}(\mathtt{u}),\mathtt{p}) \quad 0 \quad \partial_{\mathtt{u}_{|\Gamma}}\left\{\tfrac{k}{2}\beta\,\|\,\mathtt{u}_{|\Gamma}\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)\|^2 + \mathtt{I}_{\text{-}}(\,\mathtt{u}_{|\Gamma}\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)\cdot\mathtt{N}_2)\right\} \\ \partial_{\mathtt{p}}\mathtt{W}(\mathtt{e}(\mathtt{u}),\mathtt{p}) \\ \partial_\beta\mathtt{I}_{[0,1]}(\beta) + \tfrac{k}{2}\,\|\,\mathtt{u}\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)\|^2 \qquad k_s\nabla\beta \end{pmatrix}. \tag{3.10}$$

Definition 3.2.4 *For $(\mathtt{f},0) \in (H^1)^*$, $(\mathtt{g},0) \in \left(H^{1/2}\right)^*$, $\omega_s \in (L^2(\Gamma))^*$, we define the mapping $F \in V^*$ by*

$$\Big\langle\, F\,,(\mathtt{u},\mathtt{p},\beta)_V\,\Big\rangle_V := \Big\langle\,\mathtt{f}\,,\,\mathtt{u}\,\Big\rangle + \Big\langle\,\mathtt{g}\,,\,\mathtt{u}_{|\partial\Omega\setminus\Gamma}\,\Big\rangle_{\partial\Omega\setminus\Gamma} + \Big\langle\,\omega_s,\beta\,\Big\rangle_\Gamma,$$

for each $(\mathtt{u},\mathtt{p},\beta)_V \in V$.

Problem 3.2.5 *The variational problem of contact with adhesion and elastoplastic deformation is defined by:*

$$\textit{Find}\quad \mathtt{u},\,\mathtt{p},\,\beta \quad \textit{such that}\quad \partial_W\varphi(\dot{\mathtt{u}},\dot{\mathtt{p}},\dot{\beta})_W + \partial_V\psi(\mathtt{u},\mathtt{p},\beta)_V \ni F. \tag{3.11}$$

Definition 3.2.6 *A solution of the Problem 3.2.5 is a triplet $(\mathtt{u},\mathtt{p},\beta) \in H^1(0,T;V)$ satisfing* (3.11)*.*

3.3 Existence of Weak Solutions

Theorem 3.3.1 $\partial_V \psi$ *is strongly monotone.*

Before making this proof, we will prove that

Lemma 3.3.2 *The mapping* $(\mathtt{u},\mathtt{p}) \mapsto \left\langle \partial_{\mathsf{e}}\mathtt{W}(\mathtt{e}(\mathtt{u}),\mathtt{p})\,,\,\mathtt{e}(\mathtt{u})\right\rangle + \left\langle \partial_{\mathtt{p}}\mathtt{W}(\mathtt{e}(\mathtt{u}),\mathtt{p})\,,\,\mathtt{p}\right\rangle$ *is strongly monotone.*

Proof. Since $\partial_{\mathsf{e}}\mathtt{W}(\mathtt{e}(\mathtt{u}),\mathtt{p})$ and $\partial_{\mathtt{p}}\mathtt{W}(\mathtt{e}(\mathtt{u}),\mathtt{p})$ are linear for all $(\mathtt{u},\mathtt{p}) \in V$, we will prove that there exists $C > 0$ such that

$$\left\langle \partial_{\mathsf{e}}\mathtt{W}(\mathtt{e}(\mathtt{u}),\mathtt{p})\,,\,\mathtt{e}(\mathtt{u})\right\rangle + \left\langle \partial_{\mathtt{p}}\mathtt{W}(\mathtt{e}(\mathtt{u}),\mathtt{p})\,,\,\mathtt{p}\right\rangle \geq\; C\,\|(\mathtt{u},\mathtt{p})\|_V^2\,.$$

For short we will symbolize $\mathtt{e}(\mathtt{u}) = \mathtt{e}$, and $\mathtt{e}(u^i) = e^i$. From equation (1.8), $\partial_e\mathtt{W}(\mathtt{e}(u^i),p^i) = \mathcal{C}(\mathtt{e}(u^i) - p^i)$ and $\partial_p\mathtt{W}(\mathtt{e}(u^i),p^i) = -\mathcal{C}(\mathtt{e}(u^i) - p^i) + h_i p^i$. Using the Einstein summation convention,

$$\begin{aligned}
&\left\langle \partial_{\mathsf{e}}\mathtt{W}(\mathtt{e}(\mathtt{u}),\mathtt{p})\,,\,\mathtt{e}(\mathtt{u})\right\rangle + \left\langle \partial_{\mathtt{p}}\mathtt{W}(\mathtt{e}(\mathtt{u}),\mathtt{p})\,,\,\mathtt{p}\right\rangle \\
&\qquad = \left\langle \mathcal{C}(\mathtt{e}(u^i) - p^i)\,,\,\mathtt{e}(u^i) - p^i\right\rangle + h_i\|\,p^i\,\|^2 \\
&\qquad \geq 2\mu_i\left\|e^i - p^i\right\|^2 + h_i\left\|p^i\right\|^2 \\
&\qquad \geq min\left\{\mu, \tfrac{h}{2}\right\}\left[\left\|e^i - p^i\right\|^2 + \left\|p^i\right\|^2\right] + \tfrac{h}{2}\left\|p^i\right\|^2 \\
&\qquad \geq c\Big\{[\|\mathtt{e} - \mathtt{p}\| + \|\mathtt{p}\|]^2 + \|\mathtt{p}\|^2\Big\} \\
&\qquad \geq c\Big\{\|\mathtt{e}\|^2 + \|\mathtt{p}\|^2\Big\}.
\end{aligned}$$

and by Poincaré inequality over Γ (see [43, Th 1.5]),

$$c\left\{\|\mathtt{e}\|^2 + \|\mathtt{p}\|^2\right\} \geq c\left\{\,C_K\,\|\mathtt{u}\|_{H^1}^2 + \|\mathtt{p}\|^2\right\} \geq C\left\{\|\mathtt{u}\|_{H^1\times H^{1/2}}^2 + \|\mathtt{p}\|^2\right\}\,.$$

By the linearity of $\mathtt{e}$, $\partial_{\mathsf{e}}\mathtt{W}$ and $\partial_{\mathtt{p}}\mathtt{W}$,

$$\left\langle \partial_{\mathsf{e}}\mathtt{W}(\mathtt{e}(\hat{\mathtt{u}}-\mathtt{u}),\hat{\mathtt{p}} - \mathtt{p})\,,\,\mathtt{e}(\hat{\mathtt{u}}-\mathtt{u})\right\rangle + \left\langle \partial_{\mathtt{p}}\mathtt{W}(\mathtt{e}(\hat{\mathtt{u}}-\mathtt{u}),\hat{\mathtt{p}} - \mathtt{p})\,,\,\hat{\mathtt{p}} - \mathtt{p}\right\rangle$$

$$\geq C \left\{ \|\hat{\mathtt{u}}-\mathtt{u}\|^2_{H^1\times H^{1/2}} + \|\hat{\mathtt{p}}-\mathtt{p}\|^2 \right\}. \ \blacksquare$$

Proof of Theorem 3.3.1. Since $\frac{k}{2}\beta \, \| \mathtt{u}_{|\Gamma} \binom{-1}{1} \|^2$, $\mathtt{I}_{[0,1]}(\beta)$, and $\mathtt{I}_{\text{-}}(\mathtt{u}_{|\Gamma} \binom{-1}{1} \cdot \mathtt{N}_2)$ are convex, proper and l.s.c. functions, the application

$$\partial_V \left\{ \tfrac{k}{2}\beta \, \| \mathtt{u}_{|\Gamma} \tbinom{-1}{1} \|^2 + \mathtt{I}_{[0,1]}(\beta) + \mathtt{I}_{\text{-}}(\mathtt{u}_{|\Gamma} \tbinom{-1}{1} \cdot \mathtt{N}_2) \right\}$$

$$= \begin{pmatrix} 0 \quad 0 \quad 0 \quad \partial_{\mathtt{u}_{|\Gamma}} \left\{ \frac{k}{2}\beta \, \| \mathtt{u}_{|\Gamma} \binom{-1}{1} \|^2 + \mathtt{I}_{\text{-}}(\mathtt{u}_{|\Gamma} \binom{-1}{1} \cdot \mathtt{N}_2) \right\} \\ 0 \\ \partial_\beta \mathtt{I}_{[0,1]}(\beta) + \frac{k}{2} \, \| \mathtt{u} \binom{-1}{1} \|^2 \qquad 0 \end{pmatrix} \tag{3.12}$$

is monotone. By Poincaré inequality over Γ,

$$\Big\langle \, k_s \nabla \left(\beta_1 - \beta_2\right) \, , \, \nabla \left(\beta_1 - \beta_2\right) \Big\rangle_\Gamma \geq C \, \|\beta_1 - \beta_2\|^2_{H^1(\Gamma)} \, . \tag{3.13}$$

For each $\xi_1 \in \partial_V \psi(\mathtt{u},\mathtt{p},\beta)_{1_V}$, $\xi_2 \in \partial_V \psi(\mathtt{u},\mathtt{p},\beta)_{2_V}$ with $\partial_V \psi$ definied in (3.10), we can use the results of the Lemma 3.3.2 and the equations (3.12), (3.13), to obtain

$$\begin{aligned} \langle \, \xi_1 \, - \, \xi_2 \, , (\mathtt{u},\mathtt{p},\beta)_{1_V} - (\mathtt{u},\mathtt{p},\beta)_{2_V} \rangle &\geq C \left\{ \|\mathtt{u}_1 - \mathtt{u}_2\|^2_{H^1\times H^{1/2}} + \|\mathtt{p}_1 - \mathtt{p}_2\|^2 \right\} \\ &\quad + \, C \, \|\beta_1 - \beta_2\|^2_{H^1(\Gamma)} \\ &= C \, \|(\mathtt{u},\mathtt{p},\beta)_{1_V} - (\mathtt{u},\mathtt{p},\beta)_{2_V}\|^2_V \, , \qquad C > 0, \end{aligned}$$

therefore, and $\partial_V \psi$ is strongly monotone. $\blacksquare$

We want to guarantee the existence of solutions through the Theorem 1.1.25, but not directly since $\partial_W \varphi$ is not bounded. So we will prove the existence of solutions of the following differential inclusion:

$$\partial_W \, \varphi_n(\dot{\mathtt{u}}, \dot{\mathtt{p}}, \dot{\beta})_W + \partial_V \, \psi(\mathtt{u},\mathtt{p},\beta)_V \ni F, \tag{3.14}$$

$$(\mathtt{u},\mathtt{p},\beta)_V(0) = (\mathtt{u}_0, \mathtt{p}_0, \beta_0)_V \, ,$$

with $\partial_W \, \varphi_n$ bounded, where

$$\varphi_n(\mathtt{u},\mathtt{p},\beta)_W := \mathtt{R}(\mathtt{p}) + \tfrac{c_s}{2} \, \|\beta\|^2_\Gamma + \mathtt{I}^n_{\text{-}}(\beta),$$

$$\mathtt{I}^n_{\text{-}}(\beta) := \begin{cases} 0, & \text{if } \beta(\Gamma) \subset \,]-\infty, 0] \\ n\,\|\beta^+\|_\Gamma\,, & \text{otherwise} \end{cases},$$

$$\psi_n(\mathtt{u}, \mathtt{p}, \beta)_V :\equiv \psi(\mathtt{u}, \mathtt{p}, \beta)_V.$$

Later it will be proved that this solution also solves Problem 3.2.5. In the rest of the document we will symbolize

$$z(0) \;=\; z_0 = \begin{pmatrix} \mathtt{u}_0 & \mathtt{e}(\mathtt{u}_0) & \mathtt{u}_{0|_{\partial\Omega\setminus\Gamma}} & \mathtt{u}_{0|_\Gamma} \\ & & \mathtt{p}_0 & \\ & \beta_0 & \nabla\beta_0 & \end{pmatrix} \in V,$$

$$v_0 \;\in\; \begin{pmatrix} 0 & \partial_{\mathtt{e}}\mathtt{W}(\mathtt{e}(\mathtt{u}_0), \mathtt{p}_0) & 0 & \partial_{\mathtt{u}_{|\Gamma}}\left\{\frac{k}{2}\beta\,\|\,\mathtt{u}_{|\Gamma}\binom{-1}{1}\,\|^2 + \mathtt{I}_{\text{-}}(\,\mathtt{u}_{|\Gamma}\binom{-1}{1}\cdot \mathtt{N}_2)\right\}_0 \\ & & \partial_{\mathtt{p}}\mathtt{W}(\mathtt{e}(\mathtt{u}_0), \mathtt{p}_0) & \\ & \partial_\beta \mathtt{I}_{[0,1]}(\beta_0) + \frac{k}{2}\,\|\,\mathtt{u}_{0|\Gamma}\binom{-1}{1}\,\|^2 & & k_s\nabla\beta_0 \end{pmatrix},$$

and

$$F(0) = \begin{pmatrix} \mathtt{f}(0) & \mathtt{0} & \mathtt{g}(0) & \mathtt{0} \\ & & \mathtt{0} & \\ & \omega_s(0) & 0 & \end{pmatrix}.$$

Lemma 3.3.3 *If $F(0) - v_0 \in \partial_W \left\{\mathtt{R}(\mathtt{p}) + \partial\mathtt{I}^{\mathrm{k}}_{\text{-}}(\beta)\right\}_{|(p,\beta)=(0,0)}$, then for each $n \geq \mathrm{k} \in \mathbb{N}$,*

$$F(0) - v_0 \in D(\varphi_n^*).$$

Proof. Since $\partial_u\varphi = 0$, $\mathtt{f}(0) = 0$ must be assumed. Consider the representation

$$v_{0_W} = \begin{pmatrix} v_{0_1} \\ v_{0_2} \\ v_{0_3} \end{pmatrix}, \quad \text{where} \quad \begin{cases} v_{0_2} \;=\; \partial_{\mathtt{p}}\mathtt{W}(\mathtt{e}(\mathtt{u}_0), \mathtt{p}_0), \\ v_{0_3} \;\in\; \partial_\beta\mathtt{I}_{[0,1]}(\beta_0) + \frac{k}{2}\,\|\,\mathtt{u}_{0|\Gamma}\binom{-1}{1}\,\|^2 \end{cases}.$$

Since $F(0)_2 - v_{0_2} \in \partial_{\mathtt{p}}\mathtt{R}(\mathtt{0})$ and $F(0)_3 - v_{0_3} \in \partial \mathtt{I}^{\mathrm{k}}_{\text{-}}(0)$, then for all $(\mathtt{u},\mathtt{p},\beta)_W \in W$,

$$\begin{aligned} \langle F(0)_2 - v_{0_2}, \mathtt{p}\rangle &\leq \mathtt{R}(\mathtt{p}) - \mathtt{R}(\mathtt{0}), \\ \langle F(0)_3 - v_{0_3}, \beta\rangle_\Gamma &\leq \mathtt{I}^{\mathrm{k}}_{\text{-}}(\beta) - \mathtt{I}^{\mathrm{k}}_{\text{-}}(0) \leq \mathtt{I}^{n}_{\text{-}}(\beta) - \mathtt{I}^{n}_{\text{-}}(0), \end{aligned}$$

for each $n \geq \mathrm{k}$. So

$$\begin{aligned} &\langle F(0) - v_0, (\mathtt{u},\mathtt{p},\beta)_W\rangle_W - \varphi_k(\mathtt{u},\mathtt{p},\beta)_W \\ &\quad = \langle F(0)_2 - v_{0_2}, \mathtt{p}\rangle + \langle F(0)_3 - v_{0_3}, \beta\rangle_\Gamma - \mathtt{R}(\mathtt{p}) - \mathtt{I}^{n}_{\text{-}}(\beta) \\ &\quad \leq \mathtt{R}(\mathtt{0}) + \mathtt{I}^{n}_{\text{-}}(0) < \infty, \end{aligned}$$

and $F(0) - v_0 \in D(\varphi_n^*)$ for all $n \geq \mathrm{k} \in \mathbb{N}$. ■

Theorem 3.3.4 *Let $z_0 \in V$, $v_0 \in \partial\psi(z_0)$, $F \in L^1(0,T;W^*) \cap H^1(0,T;V^*)$ with the hypotheses of the Lemma 3.3.3. Then there is a weak solution to the Problem 3.2.5.*

Proof. It is not difficult to verify that $\varphi_n(\mathtt{u},\mathtt{p},\beta)_W = \mathtt{R}(\mathtt{p}) + \frac{c_s}{2}\|\beta\|^2_\Gamma + \mathtt{I}^{n}_{\text{-}}(\beta)$ and $\psi_n(\mathtt{u},\mathtt{p},\beta)_V = \psi(\mathtt{u},\mathtt{p},\beta)_V$ are propers, convex and l.s.c., functions. Further, $\partial\varphi_n$ is monotone and bounded. By the Lemma 3.3.3, $F(0) - v_0 \in D(\varphi_n^*)$ for all $n \geq \mathrm{k} \in \mathbb{N}$, and by Theorem 3.3.1 $\partial\psi$ is strongly monotone. By the Theorem 1.1.25, for each $n \geq \mathrm{k}$ there exist $z_n \in H^1(0,T;V)$, $w_n \in L^\infty(0,T;W^*)$, $v_n \in L^1(0,T;W^*) \cap L^\infty(0,T;V^*)$ that satisfy differential inclusion (3.14), this is,

$$\begin{aligned} w_n(t) + v_n(t) &= F(t), \\ w_n(t) &\in \partial\varphi_n(\dot{z}_n(t)), \\ v_n(t) &\in \partial\psi_n(z_n(t)) \qquad for \quad a.e. \quad t \in]0,T[, \\ z_n(0) &= z_0. \end{aligned}$$

If for some $m \in \mathbb{N}$ we prove that $\dot{z}_m \in D(\varphi)$, then we will have proved that z_m is a weak solution to the Problem 3.2.5.

We must assume $\{n \in \mathbb{N} : \beta_n = 0\} = \emptyset$ (otherwise $\dot{\beta}_n = 0$ and a weak solution to the problem is obtained). Therefore, $n \in \mathbb{N}$ implies $0 \neq \beta_n \leq 1$, and $\varsigma_n \geq 0$ for each $\varsigma_n \in \partial\, \mathtt{I}_{[0,1]}(\beta_n)$.

If $\#\left\{n \in \mathbb{N} :\ \dot{\beta}^{+}_{n} > 0\right\} = \infty$, then there is a sub-sequence $\dot{\beta}_{n_i}$ of $\dot{\beta}_n$, and a sub-sequence $\varsigma_{n_i} \in \partial\, \mathtt{I}_{[0,1]}(\beta_{n_i})$, such that $\partial\, \mathtt{I}_{\text{-}}^{n_i}(\dot{\beta}_{n_i}) = \frac{n_i}{\left\| \dot{\beta}_{n_i}^{+} \right\|_{\Gamma}} \dot{\beta}_{n_i}^{+}$ and $Range(\varsigma_{n_i}) \subset [0, +\infty[$. It is clear that $\dot{\beta}_{n_i}^{+} \in L^2(\Gamma)$, since $z_{n_i} \in H^1(0,T;V)$. Multiplying by $\dot{\beta}_{n_i}^{+}$ the component β of the inclusion (3.14),

$$\begin{aligned} \langle \omega_s, \dot{\beta}_{n_i}^{+} \rangle_{\Gamma} & \geq \langle c_s \dot{\beta}_{n_i} + \frac{n_i}{\left\| \dot{\beta}_{n_i}^{+} \right\|_{\Gamma}} \dot{\beta}_{n_i}^{+} \,,\, \dot{\beta}_{n_i}^{+} \rangle_{\Gamma} + \langle \varsigma_{n_i} \,,\, \dot{\beta}_{n_i}^{+} \rangle_{\Gamma} + \tfrac{k}{2} \langle \| \, \mathtt{u}_{|\Gamma} \left(\begin{smallmatrix} \text{-}1 \\ 1 \end{smallmatrix} \right) \|^2 \,,\, \dot{\beta}_{n_i}^{+} \rangle_{\Gamma} \\ & \geq \frac{n_i}{\left\| \dot{\beta}_{n_i}^{+} \right\|_{\Gamma}} \langle \dot{\beta}_{n_i}^{+} \,,\, \dot{\beta}_{n_i}^{+} \rangle_{\Gamma} \,, \end{aligned}$$

so $\|\omega_s\|_{\Gamma} \geq \langle \omega_s, \dot{\beta}_{n_i}^{+} / \left\| \dot{\beta}_{n_i}^{+} \right\|_{\Gamma} \rangle \underset{n_i \to \infty}{\longrightarrow} \infty$,which contradicts that $F \in H^1(0,T;V^*)$. For this reason it follows that

$$\# \left\{ n \in \mathbb{N} : \ \dot{\beta}_{n}^{+} > 0 \right\} < \infty,$$

and there exists $m \in \mathbb{N}$ such that $\dot{\beta}_n \leq 0$ for all $n \geq m$. This proves that $\dot{z}_m \in D(\varphi)$, and since $\partial\, \mathtt{I}_{\text{-}}^{n}(\dot{\beta}_m) \subset \partial\, \mathtt{I}_{\text{-}}(\dot{\beta}_m)$ for all $\dot{\beta}_m \leq 0$, it follows that z_m is a weak solution to the Problem 3.2.5. ■

Chapter 4

Delamination of Elastoplastic Plates

This chapter carries out the variational formulation of the adhesive contact problem (unilateral, unidirectional) between two elastoplastic plates as a rate-independent system (see Section 1.2.1). The configuration of the problem is "sandwich type" (see Figure 4.1). Unlike other works (see [23, 24]), the formulation of the model considers explicit equations deduced by Liero-Mielke, and the proof of existence and uniqueness of weak solutions is obtained from the theory of doubly nonlinear problems.

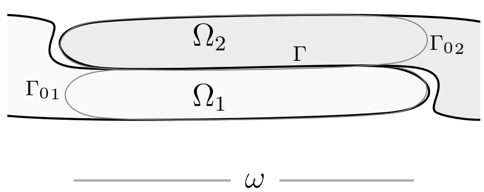

Figure 4.1: Ω_1 in adhesive contact with Ω_2 on Γ, with null displacement in Γ_{0_1} and Γ_{0_2}.

The structure of this Chapter is similar to the previous one, however, it contains more ancillary details, particularly the notation in single and double brackets introduced by the authors of the plate model.

Notation 4.0.1 *For $f : \mathbb{R}^3 \supset \omega \times [a,b] \to \mathbb{R}^{3\times 3}$, the integral of f with respect to the variable x_3 and the integral of f with respect to the variable x_3 with weight x_3, will be* *represented by*

$$[f]_0 \; : \; = \int_a^b f(.,x_3)dx_3,$$
$$[f]_1 \; : \; = \int_a^b x_3 f(.,x_3)dx_3.$$

Each symmetric matrix $A \in \mathbb{R}^{3\times 3}_{sym}$ is determined by the values a_{11}, a_{12}, a_{22}, a_{13}, a_{23}, a_{33}, therefore it can be identified if we know its submatrix (2×2), and its last column vector.

Notation 4.0.2 *For* $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \in \mathbb{R}^{3\times 3}_{sym}$, *we have the representation*

$$A = \left[\left[A^{1,2} \| A^{(3)}\right]\right],$$

where

$$A^{1,2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix},$$

and $A^{(3)} = (a_{13}, a_{23}, a_{33})^{\intercal}$.

Definition 4.0.3 *The mapping deviatoric and the spaces of deviatoric matrices are defined by*

$$\texttt{dev}\, A \; : \; = A - \frac{1}{3}\left(\texttt{Tr}\; A\right) I_{3\times 3},$$
$$\mathbb{R}^{3\times 3}_{dev} \; : \; = \{A \in \mathbb{R}^{3\times 3}_{sym} : \texttt{Tr}\; A = 0\},$$

respectively.

4.1 Abstract Formulation of the Model

Let us consider two plates Ω_1, Ω_2 with location in $\mathbb{R}^3$ given by

$$\Omega_1 = \omega \times]-1, 1[,$$
$$\Omega_2 = \omega \times]1, 3[,$$

where ω is an open and connected subset of $\mathbb{R}^2$, $\partial\omega$ is of class C^1, Ω_1 and Ω_2 in adhesive contact on $\Gamma = \omega \times \{1\}$ (development that can be generalized for n plates, see Figure 4.2).

We will assume that the adhesive material and the loads are concentrated in a proper subset of ω that does not cover the boundary $\partial\omega$; in this way it is guaranteed that there exists an open set with non-zero measure where $\beta = 0$ and $P^i = 0$. We also assume that

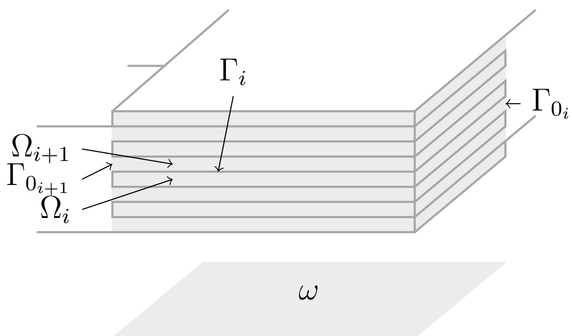

Figure 4.2: Generalization for n plates in adhesive contact.

loads per unit volume F^1, $F^2 : \mathbb{R}^3 \to \mathbb{R}^3$, and loads per unit area g^1, $g^2 : \mathbb{R}^3 \to \mathbb{R}^3$ act on each plate. As a result of that action, the rate-independent system of plates satisfy the equations formulated by Liero-Mielke[1] on elastoplastic deformation with kinematic hardening:

$$-\,\mathtt{div}\,\left(\Sigma_0\left(2E^{1,2}(u^i) - [P^{1,2}]^i_0\right)\right) - G^i_{memb}(t,\cdot) = 0 \quad \mathtt{in}\ \omega_i, \tag{4.1}$$

$$\mathtt{div}\,\mathtt{div}\,\left(\Sigma_0(\frac{2}{3}D^2u^i_3 + [P^{1,2}]^i_1)\right) - g^i_{bend}(t,\cdot) - \,\mathtt{div}\,G^i_{bend}(t,\cdot) = 0 \quad \mathtt{in}\ \omega_i, \tag{4.2}$$

$$0 \in \partial R^i(\dot{P}^i) + \mathtt{dev}\,\left[\!\left[\Sigma_0([P^{1,2}]^i \text{-} E^{1,2}(u^i) + (\mathrm{x}_3\text{-}2(\mathrm{i}\text{-}1))\,D^2u^i_3) \parallel 0\,\right]\!\right] + h_iP^i\ \mathtt{in}\ \Omega_i, \tag{4.3}$$

where $u^i : \omega \to \mathbb{R}^3 : (x_1, x_2) \mapsto (u^i_1, u^i_2, u^i_3)$ represents the displacement field of the plate[2] Ω_i, $(i = 1, 2)$, and P^i is the plastic tensor in the space $\mathbb{R}^{3\times3}_{dev}$. The rest of the

[1]The original model is defined for a plate with $x_3 \in]-1, 1[$, hence for the two plates we carry out the translation $x_3 \mapsto x_3 - 2(i-1)$, $i = 1, 2$.

[2]u^i independent of x_3.

notation corresponds to:

$$
\begin{aligned}
E^{1,2}(u) &: = \tfrac{1}{2}\left\{\mathtt{D}(u_1,u_2)+\mathtt{D}(u_1,u_2)^{\intercal}\right\},\\
[P^{1,2}]^i &: = \begin{pmatrix} P^i_{11} & P^i_{12}\\ P^i_{21} & P^i_{22}\end{pmatrix},\\
P^{(3)^i} &: = (P^i_{13},P^i_{23},P^i_{33})\\
[P]^i_0 &: = \int_{-1+2(i-1)}^{1+2(i-1)} P^i(.,x_3-2(i-1))dx_3,\\
[P]^i_1 &: = \int_{-1+2(i-1)}^{1+2(i-1)} (x_3-2(i-1))P^i(.,x_3-2(i-1))dx_3,
\end{aligned}
\tag{4.4}
$$

$$\Sigma_0(M) := \frac{2\lambda\mu}{\lambda+2\mu}(\,\mathtt{Tr}\,M)\,\mathtt{I}+2\mu M,\quad \lambda,\mu \text{ the Lamé coefficients,} \tag{4.5}$$

$$\mathtt{dev}\,M := M-\frac{1}{3}(\,\mathtt{Tr}\,M)\,\mathtt{I},\quad M\in\mathbb{R}^{3\times3}_{sym}, \tag{4.6}$$

$$
\begin{aligned}
G^i_{memb}(t,x_1,x_2) &: = [F^{1,2}(t,x_1,x_2,.)]^i_0+g^{1,2}(t,x_1,x_2,1)+g^{1,2}(t,x_1,x_2,-1),\\
G^i_{bend}(t,x_1,x_2) &: = g^{1,2}(t,x_1,x_2,-1)-g^{1,2}(t,x_1,x_2,1),\\
g^i_{bend} &: = [F^3(t,x_1,x_2,.)]^i_0+g^3(t,x_1,x_2,1)+g^3(t,x_1,x_2,-1),
\end{aligned}
$$

where $h_i > 0$ is a measure for kinematic hardening, R^i is a dissipation potential ($R(\dot{P}) = \sigma_{yield}\left\|\dot{P}\right\|$, where σ_{yield} is the yield stress, and $\|\|$ is the Euclidean norm). As a boundary condition we add $u^i_1 = u^i_2 = 0$ on $\partial\omega$, $\mathtt{D}u^i_3 = (0,0)$ on $\partial\omega$, and that there exists a nonempty section $\gamma_{i\,0} \subset \partial\omega$, such that $u^i_3 = 0$ on $\gamma_{i\,0}$, $i = 1,2$.

The displacements $u^i = (u^i_1,u^i_2,u^i_3)$ in the equations (4.1) - (4.3) correspond to points on the middle surface. On the other hand, the differential inclusions for the adhesion of the plates (1.10)-(1.11) are defined as a function of the difference of the displacements on Γ. However, the difference of the displacements on Γ can be approximated by the difference of the displacements on the middle surface. For this we consider the Kirchhoff - Love displacement field of the plate Ω_i at the points $(x,x^i_3)\in\Gamma$, $(x=(x_1,x_2))$:

$$\mathbf{u}^i_{|\Gamma}(x,x^i_3) = (u_1(x)-x^i_3\partial_{x_1}u_3(x),u_2(x)-x^i_3\partial_{x_2}u_3(x),u_3(x))^i,\ i=1,2,$$

where $x_3^2 \approx x_3^1$. Since the plates are overlap and adhered, we can assume

$$\begin{aligned}\partial_{x_1} u_3^2(x) &\approx \partial_{x_1} u_3^1(x),\\ \partial_{x_2} u_3^2(x) &\approx \partial_{x_2} u_3^1(x),\end{aligned}$$

so for x_3^1 small enough,

$$\begin{aligned}\mathbf{u}^2_{|\Gamma}(x,x_3^2) - \mathbf{u}^1_{|\Gamma}(x,x_3^1) &\approx (u_1^2(x) - u_1^1(x), u_2^2(x) - u_2^1(x), u_3^2(x) - u_3^1(x))\\ &\quad - x_3^1(\partial_{x_1} u_3^2(x) - \partial_{x_1} u_3^1(x), \partial_{x_2} u_3^2(x) - \partial_{x_2} u_3^1(x), 0)\\ &\approx u^2(x) - u^1(x).\end{aligned}$$

In conclusion, our model can consider the displacements on each point of the middle surface of the plates instead of the displacements of the points on the contact surfaces.

4.2 Weak Formulation

We define the spaces for the displacement, plastic tensor and the bonding field of the pair of plates by:

$$\mathtt{U} := \left\{ \begin{array}{ll} \mathtt{u} = (u^1, u^2) & : u^i{=}(u_1^i, u_2^i, u_3^i)^\intercal \in H^1(\omega; \mathbb{R}^{3\times 1}),\ u_3^1, u_3^2 \in H^2(\omega;\mathbb{R}) \\ = \begin{pmatrix} u_1^1 & u_1^2 \\ u_2^1 & u_2^2 \\ u_3^1 & u_3^2 \end{pmatrix} & \begin{array}{c} u_1^i = u_2^i = 0 \text{ on } \partial\omega, \quad u_3^i = 0 \text{ on } \gamma_{i\,0} \subset \partial\omega, \\ \mathtt{D}u_3^i = (0,0) \text{ on } \partial\omega, \qquad i = 1,2. \end{array} \end{array} \right\},$$

$\mathtt{P}' := \left\{ \mathtt{p} = (P^1, P^2) : P^i \in L^2(\Omega_i; \mathbb{R}^{3\times3}_{sym}),\ [P]_0^i,\ [P]_1^i \in C_c^1(\overline{\omega}; \mathbb{R}^{3\times3}_{sym}),\ i = 1,2 \right\}$, and $\mathtt{P}$ as the completion of $\mathtt{P}'$ with respect to the norm induced by the inner product

$$\left\langle P^1, \hat{P}^1 \right\rangle_{L^2(\Omega_1;\mathbb{R}^{3\times3}_{sym})} + \left\langle P^2, \hat{P}^2 \right\rangle_{L^2(\Omega_2;\mathbb{R}^{3\times3}_{sym})},$$

i.e., $\mathtt{P} = L^2(\Omega_1; \mathbb{R}^{3\times3}_{sym}) \times L^2(\Omega_2; \mathbb{R}^{3\times3}_{sym})$.

$\mathtt{B} := H^1(\Gamma; \mathbb{R})$.

We also define the product spaces:

$\mathtt{V} := \mathtt{U} \times \mathtt{P} \times \mathtt{B}$,

$\mathtt{W} := \left\{ (\mathtt{u}, \mathtt{p}, \beta) \in L^2(\omega; \mathbb{R}^{3\times2}) \times \mathtt{P} \times L^2(\Gamma; \mathbb{R}) : u_1^i = u_2^i = 0 \text{ on } \partial\omega,\ u_3^i = 0 \text{ on } \gamma_{i\,0} \right\}$.

From now on we will use Einstein's summation convention for repeated indices. The next result follows from the definitions of the spaces and Poincaré's inequality.

Lemma 4.2.1 *$\mathtt{U}$, $\mathtt{P}, \mathtt{B}, \mathtt{V}$, and $\mathtt{W}$ are Hilbert spaces under the inner product*

$$\begin{aligned}
&\langle \mathtt{u}, \tilde{\mathtt{u}} \rangle_{\mathtt{U}} = \left\langle u^i, \tilde{u}^i \right\rangle_{H^1(\omega;\mathbb{R}^{3\times 1})} + \left\langle \mathtt{D}^2 u_3^i, \mathtt{D}^2 \tilde{u}_3^i \right\rangle_{L^2(\omega;\mathbb{R}^{2\times 2})}, \\
&\langle \mathtt{p}, \tilde{\mathtt{p}} \rangle_{\mathtt{P}} = \left\langle P^i, \tilde{P}^i \right\rangle_{L^2(\Omega_i;\mathbb{R}^{3\times 3}_{sym})} = \int_{-1+2(i-1)}^{1+2(i-1)} \left\langle P^i, \tilde{P}^i \right\rangle_{L^2(\omega;\mathbb{R}^{3\times 3}_{sym})} (x_3 - 2(i-1)) \quad dx_3, \\
&\left\langle \beta, \tilde{\beta} \right\rangle_{\mathtt{B}} = \left\langle \beta, \tilde{\beta} \right\rangle_{H^1(\Gamma)}, \\
&\left\langle (\mathtt{u}, \mathtt{p}, \beta), (\tilde{\mathtt{u}}, \tilde{\mathtt{p}}, \tilde{\beta}) \right\rangle_{\mathtt{V}} = \langle \mathtt{u}, \tilde{\mathtt{u}} \rangle_{\mathtt{U}} + \langle \mathtt{p}, \tilde{\mathtt{p}} \rangle_{\mathtt{P}} + \left\langle \beta, \tilde{\beta} \right\rangle_{\mathtt{B}}, \\
&\left\langle (\mathtt{u}, \mathtt{p}, \beta), (\tilde{\mathtt{u}}, \tilde{\mathtt{p}}, \tilde{\beta}) \right\rangle_{\mathtt{W}} = \langle \mathtt{u}, \tilde{\mathtt{u}} \rangle_{L^2(\omega;\mathbb{R}^{3\times 2})} + \langle \mathtt{p}, \tilde{\mathtt{p}} \rangle_{\mathtt{P}} + \left\langle \beta, \tilde{\beta} \right\rangle_{L^2(\Gamma;\mathbb{R})},
\end{aligned}$$

Also, $\mathtt{V}$ is dense in $\mathtt{W}$ with compact immersion.

Lemma 4.2.2 *If $\mathtt{u}, \hat{\mathtt{u}} \in C^2(\omega;\mathbb{R}^{3\times 2})$, $u_3^i, \tilde{u}_3^i \in C^4(\omega;\mathbb{R})$, $i = 1, 2$, $p \in \mathtt{P}'$, $(\mathtt{u}, \mathtt{p})$ verifying the boundary conditions given in $\mathtt{V}$, then*

1. $\left\langle -\mathtt{div}\, \Sigma_0(2\mathtt{E}^{1,2}(u^i) - [P^{1,2}]_0^i), \begin{pmatrix} \tilde{u}_1^i \\ \tilde{u}_2^i \end{pmatrix} \right\rangle_{L^2(\omega)^2} = \left\langle \Sigma_0(2\mathtt{E}^{1,2}(u^i) - [P^{1,2}]_0^i) : \mathtt{E}^{1,2}(\tilde{u}^i) \right\rangle_{L^2(\omega)^{2\times 2}}$

2. $\left\langle \mathtt{div}\,\mathtt{div}\, \Sigma_0(\tfrac{2}{3}\mathtt{D}^2 u_3^i - [P^{1,2}]_1^i), \tilde{u}_3^i \right\rangle_{L^2(\omega)} = \left\langle \Sigma_0(\tfrac{2}{3}\mathtt{D}^2 u_3^i - [P^{1,2}]_1^i) : \mathtt{D}^2 \tilde{u}_3^i \right\rangle_{L^2(\omega)^{2\times 2}}$

Proof. 1. Suppose that $\mathtt{u}, \hat{\mathtt{u}} : \omega \to \mathbb{R}^{3\times 2}$, $P^i : \Omega_i \to \mathbb{R}^{3\times 3}$ verify the hypotheses of theorem. Let's define $A^i := \Sigma_0(2\mathtt{E}^{1,2}(u^i) - [P^{1,2}]_0^i)$. Then $\mathtt{div}\, A^i = \mathtt{div}\, \Sigma_0(2\mathtt{E}^{1,2}(u^i) - [P^{1,2}]_0^i) : \omega \to \mathbb{R}^2$. By integration by parts and theorem of divergence,

$$\begin{aligned}
\int_\omega \mathtt{div}\, A^i \cdot \begin{pmatrix} \tilde{u}_1^i \\ \tilde{u}_2^i \end{pmatrix} &= \int_\omega \mathtt{div} \left\{ A^i \begin{pmatrix} \tilde{u}_1^i \\ \tilde{u}_2^i \end{pmatrix} \right\} - \int_\omega A^i : \mathtt{D} \begin{pmatrix} \tilde{u}_1^i \\ \tilde{u}_2^i \end{pmatrix} \\
&= \int_{\partial\omega} A^i \begin{pmatrix} \tilde{u}_1^i \\ \tilde{u}_2^i \end{pmatrix} \cdot \mathtt{n}_i - \int_\omega A^i : \mathtt{D} \begin{pmatrix} \tilde{u}_1^i \\ \tilde{u}_2^i \end{pmatrix},
\end{aligned}$$

where $\mathtt{n}_i$ is the normal vector to $\partial\omega$. Due to boundary conditions on $\partial\omega$, $\int_{\partial\omega} A^i \begin{pmatrix} \tilde{u}_1^i \\ \tilde{u}_2^i \end{pmatrix} \cdot$

$\mathtt{n}_i = 0$. So

$$\int_\omega \mathtt{div}\,\Sigma_0(2\mathtt{E}^{1,2}(u^i) - [P^{1,2}]_0^i) \cdot \begin{pmatrix} \tilde{u}_1^i \\ \tilde{u}_2^i \end{pmatrix} = \int_\omega A^i : \mathtt{D} \begin{pmatrix} \tilde{u}_1^i \\ \tilde{u}_2^i \end{pmatrix}.$$

Since $\Sigma_0(2\mathtt{E}^{1,2}(u^i) - [P^{1,2}]_0^i)$ is symmetric, then

$$\Sigma_0(2\mathtt{E}^{1,2}(u^i) - [P^{1,2}]_0^i) : \mathtt{D} \begin{pmatrix} \tilde{u}_1^i \\ \tilde{u}_2^i \end{pmatrix} = \Sigma_0(2\mathtt{E}^{1,2}(u^i) - [P^{1,2}]_0^i) : \mathtt{D} \begin{pmatrix} \tilde{u}_1^i \\ \tilde{u}_2^i \end{pmatrix}^\top,$$

for that

$$\begin{aligned} \left\langle \mathtt{-div}\,\Sigma_0(2\mathtt{E}^{1,2}(u^i) - [P^{1,2}]_0^i), \begin{pmatrix} \tilde{u}_1^i \\ \tilde{u}_2^i \end{pmatrix} \right\rangle_{L^2(\omega;\mathbb{R}^2)} &= \int_\omega A^i : \tfrac{1}{2}\left(\mathtt{D} + \mathtt{D}^\top\right) \begin{pmatrix} \tilde{u}_1^i \\ \tilde{u}_2^i \end{pmatrix} \\ &= \left\langle \Sigma_0(2\mathtt{E}^{1,2}(u^i) - [P^{1,2}]_0^i) : \mathtt{E}^{1,2}(\tilde{u}^i) \right\rangle_{L^2(\omega;\mathbb{R}^{2\times 2})}. \end{aligned}$$

2. Let's define $B^i = \Sigma_0(\frac{2}{3}\mathtt{D}^2 u_3^i - [P^{1,2}]_1^i)$, by integration by parts and theorem of divergence,

$$\begin{aligned} \left\langle \mathtt{div}\,\mathtt{div}\,B^i, \tilde{u}_3^i \right\rangle_{L^2(\omega;\mathbb{R})} &= \int_\omega \mathtt{div}\left\{\mathtt{div}\,B^i\,\tilde{u}_3^i\right\} - \int_\omega \mathtt{div}\,B^i\,\mathtt{D}\tilde{u}_3^i \\ &= \int_{\partial\omega} \left\{\mathtt{div}\,B^i\,\tilde{u}_3^i\right\} \cdot \mathtt{n}_i - \int_\omega \mathtt{div}\left\{B^i\,\mathtt{D}\tilde{u}_3^i\right\} + \int_\omega \left\{B^i\,\mathtt{D}^2\tilde{u}_3^i\right\} \\ &= \int_{\partial\omega} \left\{\mathtt{div}\,B^i\,\tilde{u}_3^i\right\} \cdot \mathtt{n}_i - \int_{\partial\omega} B^i\,\mathtt{D}\tilde{u}_3^i \cdot \mathtt{n}_i + \int_\omega \left\{B^i\,\mathtt{D}^2\tilde{u}_3^i\right\} \end{aligned}$$

and by the boundary conditions in $\mathtt{V}$, we obtain

$$\left\langle \mathtt{div}\,\mathtt{div}\,\Sigma_0(\tfrac{2}{3}\mathtt{D}^2 u_3^i - [P_i^{1,2}]_1), \tilde{u}_3^i \right\rangle_{L^2(\omega)} = \left\langle \Sigma_0\left(\tfrac{2}{3}\mathtt{D}^2 u_3^i - [P_i^{1,2}]_1\right) : \mathtt{D}^2\tilde{u}_3^i \right\rangle_{L^2(\omega)^{2\times 2}}. \quad \blacksquare$$

Notation 4.2.3 *To shorten the writing of the calculations, we will use the following notations:*

$$\begin{aligned} C^i \;&:= \mathtt{dev}\left[\left[\Sigma_0(\left[P^{1,2}\right]^i - \mathtt{E}^{1,2}(u^i) + x_3 D^2 u_3^i) \parallel 0\right]\right] + h_i P^i, \\ A \;&:= (\left[\left[A^1\|0\right]\right], \left[\left[A^2\|0\right]\right]), \quad B := (B^1, B^2), \quad C := (C^1, C^2), \\ \mathtt{u}_3 \;&:= (u_3^1, u_3^2), \quad \mathtt{D}^2\mathtt{u}_3 = (\mathtt{D}^2 u_3^1, \mathtt{D}^2 u_3^2), \\ \mathtt{E}^{1,2}(\mathtt{u}) \;&:= (\left[\left[\mathtt{E}^{1,2}(u^1)\|0\right]\right], \left[\left[\mathtt{E}^{1,2}(u^2)\|0\right]\right]), \end{aligned}$$

$$\mathtt{F} \ : \ = \begin{pmatrix} [f_1]_0^1 + g_1^1{}_{x_3=-1} & [f_1]_0^2 + g_1^2{}_{x_3=1} \\ [f_2]_0^1 + g_2^1{}_{x_3=-1} & [f_2]_0^2 + g_2^2{}_{x_3=1} \\ 0 & 0 \end{pmatrix},$$

$$\mathtt{G} \ : \ = \left([f_3]_0^1 + g_3^1{}_{x_3=-1} + \mathtt{div}(g_1^1, g_2^1)^{\mathsf{T}}_{x_3=-1}\, ,\, [f_3]_0^2 + g_3^2{}_{x_3=1} + \mathtt{div}(g_1^2, g_2^2)^{\mathsf{T}}_{x_3=1}\right).$$

With these notations, we have

Lemma 4.2.4 *If* $\mathtt{u} \in L^1(0,T;\mathtt{U})$, $\mathtt{p} \in C^1(0,T;\mathtt{P})$ *satisfy* (4.1) - (4.3), $\mathtt{q} = (q^1, q^2) \in \left(\partial\,\mathtt{R}^1\,(\dot{P}^1), \partial\,\mathtt{R}^2\,(\dot{P}^2)\right)$, *then for all* $(\hat{\mathtt{u}}, \hat{\mathtt{p}}) \in L^1(0,T;\mathtt{U}\times\mathtt{P})$, ,

$$\left\langle A\,(\mathtt{u},\mathtt{p})\,,\,\mathtt{E}^{1,2}(\hat{\mathtt{u}})\right\rangle_{L^2(\omega;\mathbb{R}^{2\times 2})^2} = \left\langle \mathtt{F}\,,\,\hat{\mathtt{u}}\right\rangle_{L^2(\omega;\mathbb{R}^3)^2} \tag{4.7}$$

$$\left\langle B\,(\mathtt{u},\mathtt{p})\,,\,\mathtt{D}^2(\hat{\mathtt{u}}_3)\right\rangle_{L^2(\omega;\mathbb{R}^{2\times 2})^2} = \left\langle \mathtt{G}\,,\,\hat{\mathtt{u}}_3\right\rangle_{L^2(\omega;\mathbb{R})^2} \tag{4.8}$$

$$\left\langle \mathtt{q}\,,\,\hat{\mathtt{p}}\right\rangle_{L^2(\Omega_1)^{3\times 3}\times L^2(\Omega_2)^{3\times 3}} + \left\langle C(\mathtt{u},\mathtt{p})\,,\,\hat{\mathtt{p}}\right\rangle_{L^2(\Omega_1)^{3\times 3}\times L^2(\Omega_2)^{3\times 3}} = 0 \tag{4.9}$$

Proof. Let us consider $\mathtt{p} \in C^1(0,T;\mathtt{P}')$. Multiplying the equation (4.1) by $\begin{pmatrix} \tilde{u}_1^i \\ \tilde{u}_2^i \end{pmatrix}$, the Equation (4.2) by $\tilde{u}_3^i$, the Equation (4.3) by $\tilde{P}^i$, and using the identities of Lemma 4.2.2, the Notation 4.2.3, and the density of $\mathtt{P}'$ in $\mathtt{P}$, the result is concluded. ■

Notation 4.2.5 *To simplify the notation, we will use* $\langle .,. \rangle$ *to indicate any of the inner products* $\langle .,. \rangle_{L^2(\omega;\mathbb{R}^{m\times n})^s}$, *or any dual pairing* $\langle .,. \rangle_{X^*\times X}$.

Remark 4.2.6 *Since* $\mathtt{Tr}P^i = \mathtt{Tr}\hat{P}^i = 0$, *then*

$$\int_{-1}^{1} \left\langle \mathtt{Tr}\Sigma_0\left(x_3 D^2 v_3^i + \left[P^{1,2}\right]^i - \mathtt{E}^{1,2}(u^i)\right)\mathtt{I}_{3\times 3}, \hat{P}^i\right\rangle$$
$$= \int_{-1}^{1} \left\langle \mathtt{Tr}\Sigma_0\left(x_3 D^2 v_3^i + \left[P^{1,2}\right]^i - \mathtt{E}^{1,2}(u^i)\right), \mathtt{Tr}\hat{P}^i\right\rangle = 0,$$

for such,

$$\begin{aligned}
&\int_{-1}^{1} \left\langle \texttt{dev}\left[\left[\Sigma_0\left(x_3 D^2 v_3^i + \left[P^{1,2}\right]^i - \mathtt{E}^{1,2}(u^i)\right) \| 0\right]\right], \hat{P}^i \right\rangle \\
&= \int_{-1}^{1} \left\langle \Sigma_0 \left(x_3 D^2 v_3^i + \left[P^{1,2}\right]^i - \mathtt{E}^{1,2}(u^i)\right), \left[\hat{P}^{1,2}\right]^i \right\rangle \\
&= \int_{-1}^{1} \left\{ \left\langle \Sigma_0 D^2 v_3, x_3 \left[\hat{P}^{1,2}\right]^i \right\rangle + \left\langle \Sigma_0 \left[P^{1,2}\right]^i - \Sigma_0 \mathtt{E}^{1,2}(u^i), \left[\hat{P}^{1,2}\right]^i \right\rangle \right\} \\
&= \left\langle \Sigma_0 D^2 v_3, \left[\hat{P}^{1,2}\right]_1^i \right\rangle + \int_{-1}^{1} \left\langle \Sigma_0 \left[P^{1,2}\right]^i, \left[\hat{P}^{1,2}\right]^i \right\rangle - \left\langle \Sigma_0 \mathtt{E}^{1,2}(u^i), \left[\hat{P}^{1,2}\right]_0^i \right\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\left\langle C(\mathtt{u},\mathtt{p}), \hat{\mathtt{p}} \right\rangle &= \left\langle \Sigma_0 D^2 v_3^i, \left[\hat{P}^{1,2}\right]_1^i \right\rangle - \left\langle \Sigma_0 \mathtt{E}^{1,2}(u^i), \left[\hat{P}^{1,2}\right]_0^i \right\rangle \\
&\quad + \int_{-1}^{1} \left\{ \left\langle \Sigma_0 \left[P^{1,2}\right]^i, \left[\hat{P}^{1,2}\right]^i \right\rangle + h^i \left\langle P^i, \hat{P}^i \right\rangle \right\} \qquad (4.10)
\end{aligned}$$

Remark 4.2.7 *Since* $\mathtt{N}_2 = (0,0,-1)^\top$, *then* $\mathtt{u}\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right) \cdot \mathtt{N}_2 = u_3^1 - u_3^2$.

Lemma 4.2.8 *If* $\mathtt{u} \in L^1(0,T;\mathtt{U})$, $\beta \in C^1(0,T;\mathtt{B})$ *satisfy* (1.10)-(1.13), $\varrho \in \partial\, \mathtt{I}_{\text{-}}(\mathtt{u}\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right) \cdot \mathtt{N}_2)$, $\alpha \in \partial\, \mathtt{I}_{\text{-}}(\dot{\beta})$, $\varsigma \in \partial\, \mathtt{I}_{[0,1]}(\beta)$, *then for all* $(\hat{\mathtt{u}},\hat{\beta}) \in L^1(0,T;\mathtt{U}\times\mathtt{B})$,

$$k\left\langle \beta\, \mathtt{u}\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right), \hat{\mathtt{u}}\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right) \right\rangle_\Gamma + \int_\Gamma \varrho\, \hat{\mathtt{u}}\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right) \cdot \mathtt{N}_2 \, ds = 0 \qquad (4.11)$$

$$\left\langle \alpha + \varsigma, \hat{\beta} \right\rangle_\Gamma + k_s \left\langle \nabla\beta, \nabla\hat{\beta} \right\rangle_\Gamma + \tfrac{k}{2} \left\langle \|\mathtt{u}\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)\|^2, \hat{\beta} \right\rangle_\Gamma = \left\langle w_s, \hat{\beta} \right\rangle_\Gamma \qquad (4.12)$$

Proof. As σ^i we will consider the stress tensor $\left[\left[\Sigma_0(2\mathtt{E}^{1,2}(u^i) - [P^{1,2}]_0^i)\|0\right]\right]$ (see [36] p. 18), as a result of that we obtain $\sigma^i \mathtt{N}_i = 0$. Multiplying the first Equation of (1.11) by $\hat{u}^1$, the second equation by $\hat{u}^2$, and integrating over Γ, we obtain the equation (4.11). The Equation (4.12) results from integrating over Γ the product of the Equation (1.10) by $\hat{\beta}$, and from using the boundary condition (1.12) on the term $\int_\omega (k_s \Delta_s \beta)\, (\hat{\beta}) = \int_{\partial\omega} (k_s \nabla_s \beta)\, (\hat{\beta}) \cdot n - \int_\omega (k_s \nabla_s \beta)\, (\nabla_s \hat{\beta})$. ■

Problem 4.2.9 *The weak formulation of the problem of unilateral contact with unidirectional adhesion and elastoplastic deformation of plates with hardening is defined as: find* $(\mathtt{u},\mathtt{p},\beta)$ *that satisfies* (4.7)-(4.9), (4.11)-(4.12).

We will proceed to rewrite the Problem 4.2.9 as a doubly non-linear problem.

Definition 4.2.10 *Let* $(\hat{\mathtt{u}}, \hat{\mathtt{p}}, \hat{\beta}) \in V$. *We define by* $\psi : \mathtt{V} \to]-\infty, \infty]$, $\psi_1 : \mathtt{U} \times \mathtt{B} \to]-\infty, \infty]$, $\psi_{(\hat{\mathtt{u}},\hat{\mathtt{p}})} : \mathtt{U} \times \mathtt{P} \to]-\infty, \infty]$, *and* $\varphi : \mathtt{W} \to]-\infty, \infty]$ *the functional*

$$\begin{aligned}
\varphi(\mathtt{u}, \mathtt{p}, \beta)_W &= \mathtt{R}(\mathtt{p}) + \mathtt{I}_{\text{-}}(\beta), \\
\psi_1(\mathtt{u}, \beta) &= \tfrac{k_s}{2} \|\nabla\beta\|^2 + \tfrac{k}{2}\beta \left\|\mathtt{u}\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)\right\|^2 + \mathtt{I}_{[0,1]}(\beta) + \mathtt{I}_{\text{-}}(u_3^1 - u_3^2), \\
\psi_{(\hat{\mathtt{u}},\hat{\mathtt{p}})}(\mathtt{u}, \mathtt{p}) &= \left\langle A(\hat{\mathtt{u}}, \hat{\mathtt{p}}), \mathtt{E}^{1,2}(\mathtt{u}) \right\rangle + \left\langle B(\hat{\mathtt{u}}, \hat{\mathtt{p}}), \mathtt{D}^2\mathtt{u}_3 \right\rangle + \langle C(\hat{\mathtt{u}}, \hat{\mathtt{p}}), \mathtt{p} \rangle \\
\psi(\mathtt{u}, \mathtt{p}, \beta)_V &= \psi_1(\mathtt{u}, \beta) + \psi_{(\hat{\mathtt{u}},\hat{\mathtt{p}})}(\mathtt{u}, \mathtt{p})
\end{aligned}$$

and their respective subdifferentials $\partial_W \varphi : \mathtt{W} \rightrightarrows \mathtt{W}^*$, *and* $\partial_V \psi : \mathtt{V} \rightrightarrows \mathtt{V}^*$ *by*

$$\partial_W \varphi(\hat{\mathtt{u}}, \hat{\mathtt{p}}, \hat{\beta})_W = \begin{pmatrix} 0 \\ \partial_{\mathtt{p}} \mathtt{R}(\hat{\mathtt{p}}) \\ \partial_\beta \mathtt{I}_{\text{-}}(\hat{\beta}) \end{pmatrix},$$

$$\partial_V \psi(\hat{\mathtt{u}}, \hat{\mathtt{p}}, \hat{\beta})_V = \begin{pmatrix} \partial_{\mathtt{u}}\psi_1 & \partial_{\mathtt{E}^{1,2}(\mathtt{u})}\psi & \partial_{\mathtt{Du}_3}\psi & \partial_{\mathtt{D}^2\mathtt{u}_3}\psi \\ & \partial_{\mathtt{p}}\psi & & \\ & \partial_\beta\psi & \partial_{\nabla\beta}\psi & \end{pmatrix} (\hat{\mathtt{u}}, \hat{\mathtt{p}}, \hat{\beta})$$

$$= \begin{pmatrix} \partial_{\mathtt{u}}\psi_1(\hat{\mathtt{u}}, \hat{\beta}) & A(\hat{\mathtt{u}}, \hat{\mathtt{p}}) & 0 & B(\hat{\mathtt{u}}, \hat{\mathtt{p}}) \\ & C(\hat{\mathtt{u}}, \hat{\mathtt{p}}) & & \\ \partial_\beta \mathtt{I}_{[0,1]}(\hat{\beta}) + \frac{k}{2}\left\|\hat{\mathtt{u}}\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)\right\|^2 & & & k_s\nabla\hat{\beta} \end{pmatrix},$$

where $\left\langle \partial_{\mathtt{u}}\psi_1(\hat{\mathtt{u}}, \hat{\beta}), u \right\rangle = \left\langle k\hat{\beta}(\hat{u}^2 - \hat{u}^1), (u^2 - u^1) \right\rangle_{L^2(\omega;\mathbb{R}^3)} + \langle \partial_{\mathtt{u}} \mathtt{I}_{\text{-}}(u_3^2 - u_3^1), (u^2 - u^1) \rangle_{L^2(\omega;\mathbb{R}^3)}$.

Definition 4.2.11 *For* $\mathtt{F} \in H^1(\omega; \mathbb{R}^{2\times 2})^*$, $\mathtt{G} \in H^2(\omega; \mathbb{R}^2)^*$, $\omega_s \in (L^2(\Gamma))^*$, *we define the mapping* $F \in \mathtt{V}^*$ *by*

$$\left\langle F, (\mathtt{u}, \mathtt{p}, \beta)_V \right\rangle_V = \left\langle \mathtt{F}, \mathtt{u} \right\rangle + \left\langle \mathtt{G}, \mathtt{u}_3 \right\rangle + \left\langle 0, \mathtt{p} \right\rangle + \left\langle \omega_s, \beta \right\rangle_\Gamma$$

for each $(\mathtt{u}, \mathtt{p}, \beta)_V \in \mathtt{V}$.

Problem 4.2.12 *The weak problem of contact with adhesion and elastoplastic deformation is defined by:*

$$\textit{Find} \quad \mathtt{u},\, \mathtt{p},\, \beta \quad \textit{such that} \quad \partial_W \varphi(\dot{\mathtt{u}}, \dot{\mathtt{p}}, \dot{\beta})_W + \partial_V \psi(\mathtt{u}, \mathtt{p}, \beta)_V \ni F \tag{4.13}$$

Definition 4.2.13 *A weak solution of the Problem 4.2.12 is a triplet* $(\mathtt{u}, \mathtt{p}, \beta) \in H^1(0,T;\mathtt{V})$ *satisfing* (4.13).

4.3 Existence of Weak Solutions

Theorem 4.3.1 *If* $h_i > 14\left(\frac{2\lambda_i\mu_i}{\lambda_i+2\mu_i} + 2\mu_i\right)$, $i = 1, 2$, *then the mapping* $\partial\psi_{(\hat{\mathtt{u}},\hat{\mathtt{p}})}$ *is strongly monotone.*

Proof. Let's define α_1, α_2 as $\alpha_i = 8/\left(\frac{h_i}{\frac{2\lambda_i\mu_i}{\lambda_i+2\mu_i}+2\mu_i} - 1\right)$, then $0 < \alpha_i < \frac{8}{13} < \frac{2}{3}$, and

$$\begin{aligned}
\left(1+\tfrac{8}{\alpha_i}\right)\int_{-1}^{1}\left\langle \Sigma_0\left[P^{1,2}\right]^i, \left[P^{1,2}\right]^i\right\rangle &= \left(1+\tfrac{8}{\alpha_i}\right)\int_{-1}^{1}\left\{\tfrac{2\lambda_i\mu_i}{\lambda_i+2\mu_i}\left\langle \mathtt{Tr}\left[P^{1,2}\right]^i, \mathtt{Tr}\left[P^{1,2}\right]^i\right\rangle\right. \\
&\qquad\qquad \left. +2\mu_i\left\langle \left[P^{1,2}\right]^i, \left[P^{1,2}\right]^i\right\rangle\right\} \\
&\le \left(1+\tfrac{8}{\alpha_i}\right)\int_{-1}^{1}\left(\tfrac{2\lambda_i\mu_i}{\lambda_i+2\mu_i}+2\mu_i\right)\left\langle \left[P^{1,2}\right]^i, \left[P^{1,2}\right]^i\right\rangle \\
&= \frac{h_i}{\frac{2\lambda_i\mu_i}{\lambda_i+2\mu_i}+2\mu_i}\left(\tfrac{2\lambda_i\mu_i}{\lambda_i+2\mu_i}+2\mu_i\right)\int_{-1}^{1}\left\langle \left[P^{1,2}\right]^i, \left[P^{1,2}\right]^i\right\rangle \\
&= h_i\int_{-1}^{1}\left\langle \left[P^{1,2}\right]^i, \left[P^{1,2}\right]^i\right\rangle .
\end{aligned} \tag{4.14}$$

We will prove that $\left\langle \partial\psi_{(\mathtt{u},\mathtt{p})}, (\mathtt{u},\mathtt{p})\right\rangle \ge c\left\|(\mathtt{u},\mathtt{p})\right\|^2_{U\times P}$ using the result (4.14) and the fact that $\langle \Sigma_0 M, M\rangle \ge 0$, $M \in \mathbb{R}^{2\times 2}$ in the following chain of inequalities:

$$\begin{aligned}
&\left\langle \partial\psi_{(\mathtt{u},\mathtt{p})}, (\mathtt{u},\mathtt{p})\right\rangle \\
&\quad = \left\langle \Sigma_0\left(2\mathtt{E}^{1,2}(u^i) - \left[P^{1,2}\right]^i_0\right), \mathtt{E}^{1,2}(u^i)\right\rangle + \left\langle \Sigma_0\left(\tfrac{2}{3}\mathtt{D}^2 v^i_3 + \left[P^{1,2}\right]^i_1\right), \mathtt{D}^2 u^i_3\right\rangle \\
&\qquad\quad + \left\langle \Sigma_0 \mathtt{D}^2 u^i_3, \left[P^{1,2}\right]^i_1\right\rangle - \left\langle \Sigma_0\mathtt{E}^{1,2}(u^i), \left[P^{1,2}\right]^i_0\right\rangle \\
&\qquad\qquad + \int_{-1}^{1}\left\{\left\langle \Sigma_0\left[P^{1,2}\right]^i, \left[P^{1,2}\right]^i\right\rangle + h_i\left\langle P^i, P^i\right\rangle\right\}
\end{aligned}$$

$$= \left\langle \Sigma_0 \left(2\mathtt{E}^{1,2}(u^i) - \left[P^{1,2}\right]_0^i \right), \mathtt{E}^{1,2}(u^i) \right\rangle - \left\langle \Sigma_0 \mathtt{E}^{1,2}(u^i), \left[P^{1,2}\right]_0^i \right\rangle + \left\langle \Sigma_0 \left(\tfrac{2}{3}\mathtt{D}^2 u_3^i + 2\left[P^{1,2}\right]_1^i \right), \mathtt{D}^2 u_3^i \right\rangle + \int_{-1}^{1} \left\{ \left\langle \Sigma_0 \left[P^{1,2}\right]^i, \left[P^{1,2}\right]^i \right\rangle + h_i \left\langle P^i, P^i \right\rangle \right\}$$

$$= \left\langle \Sigma_0 \mathtt{E}^{1,2}(u^i), \mathtt{E}^{1,2}(u^i) \right\rangle + \left\langle \Sigma_0 \left(\mathtt{E}^{1,2}(u^i) - \left[P^{1,2}\right]_0^i \right), \mathtt{E}^{1,2}(u^i) - \left[P^{1,2}\right]_0^i \right\rangle + \left(\tfrac{2}{3} - \alpha_i\right) \left\langle \Sigma_0 \mathtt{D}^2 u_3^i, \mathtt{D}^2 u_3^i \right\rangle + \alpha_i \left\langle \Sigma_0 \left(\mathtt{D}^2 u_3^i + \tfrac{2}{\alpha_i}\left[P^{1,2}\right]_1^i \right), \mathtt{D}^2 u_3^i \right\rangle - \left\langle \Sigma_0 \left[P^{1,2}\right]_0^i, \left[P^{1,2}\right]_0^i \right\rangle + \int_{-1}^{1} \left\{ \left\langle \Sigma_0 \left[P^{1,2}\right]^i, \left[P^{1,2}\right]^i \right\rangle + h_i \left\langle P^i, P^i \right\rangle \right\}$$

$$\geq \left\langle \Sigma_0 \mathtt{E}^{1,2}(u^i), \mathtt{E}^{1,2}(u^i) \right\rangle + \left(\tfrac{2}{3} - \alpha_i\right) \left\langle \Sigma_0 \mathtt{D}^2 u_3, \mathtt{D}^2 u_3 \right\rangle + \alpha_i \left\langle \Sigma_0 \left(\mathtt{D}^2 u_3^i + \tfrac{2}{\alpha_i}\left[P^{1,2}\right]_1^i \right), \mathtt{D}^2 u_3^i + \tfrac{2}{\alpha_i}\left[P^{1,2}\right]_1^i \right\rangle - \left\langle \Sigma_0 \left[P^{1,2}\right]_0^i, \left[P^{1,2}\right]_0^i \right\rangle - \tfrac{4}{\alpha_i} \left\langle \Sigma_0 \left[P^{1,2}\right]_1^i, \left[P^{1,2}\right]_1^i \right\rangle + \int_{-1}^{1} \left\{ \left\langle \Sigma_0 \left[P^{1,2}\right]^i, \left[\hat{P}^{1,2}\right]^i \right\rangle + h_i \left\langle P^i, P^i \right\rangle \right\}$$

$$\geq \left\langle \Sigma_0 \mathtt{E}^{1,2}(u^i), \mathtt{E}^{1,2}(u^i) \right\rangle + \left(\tfrac{2}{3} - \alpha_i\right) \left\langle \Sigma_0 \mathtt{D}^2 u_3^i, \mathtt{D}^2 u_3^i \right\rangle - \left\langle \Sigma_0 \left[P^{1,2}\right]_0^i, \left[P^{1,2}\right]_0^i \right\rangle - \tfrac{8}{\alpha_i} \int_{-1}^{1} \left\langle \Sigma_0 \left[P^{1,2}\right]^i, \left[P^{1,2}\right]^i \right\rangle + \int_{-1}^{1} \left\{ \left\langle \Sigma_0 \left[P^{1,2}\right]^i, \left[P^{1,2}\right]^i \right\rangle + h_i \left\langle P^i, P^i \right\rangle \right\}$$

$$\geq \left\langle \Sigma_0 \mathtt{E}^{1,2}(u^i), \mathtt{E}^{1,2}(u^i) \right\rangle + \left(\tfrac{2}{3} - \alpha_i\right) \left\langle \Sigma_0 \mathtt{D}^2 u_3^i, \mathtt{D}^2 u_3^i \right\rangle - \int_{-1}^{1} \left\langle \Sigma_0 \left[P^{1,2}\right]^i, \left[P^{1,2}\right]^i \right\rangle - \tfrac{8}{\alpha_i} \int_{-1}^{1} \left\langle \Sigma_0 \left[P^{1,2}\right]^i, \left[P^{1,2}\right]^i \right\rangle + h_i \int_{-1}^{1} \left\langle P^i, P^i \right\rangle$$

$$\geq c\left\{\left\langle \mathtt{E}^{1,2}(u^i), \mathtt{E}^{1,2}(u^i)\right\rangle + \left\langle \Sigma_0 \mathtt{D}^2 u_3^i, \mathtt{D}^2 u_3^i\right\rangle + \int_{-1}^{1}\left\langle P^i, P^i\right\rangle\right\},$$

$$c = \min_{i=1,2}\left\{1,\ \tfrac{2}{3}-\alpha_i,\ h_i - \left(1+\tfrac{8}{\alpha_i}\right)\left(\tfrac{2\lambda_i\mu_i}{\lambda_i+2\mu_i}+2\mu_i\right)\right\},$$

and by Poincaré inequality, $\left\langle \partial\psi_{(\mathtt{u},\mathtt{p})}, (\mathtt{u},\mathtt{p})\right\rangle \geq c\left\{\|\mathtt{u}\| + \|\mathtt{p}\|\right\}$. ■

Corollary 4.3.2 *If* $h_i > 14\left(\frac{2\lambda_i\mu_i}{\lambda_i+2\mu_i}+2\mu_i\right)$, $i=1,2$, *then* $\partial\psi$ *is strongly monotone.*

Proof. Since $\frac{k}{2}\beta\,\|\,\mathtt{u}\binom{-1}{1}\,\|^2$, $\mathtt{I}_{[0,1]}(\beta)$, and $\mathtt{I}_{\text{-}}(\,\mathtt{u}\binom{-1}{1}\,\cdot\mathtt{N}_2)$ are convex, proper and l.s.c. functions,

$$\partial_V\left\{\ \tfrac{k}{2}\beta\,\|\,\mathtt{u}\tbinom{-1}{1}\,\|^2 + \mathtt{I}_{[0,1]}(\beta) + \mathtt{I}_{\text{-}}(\mathtt{u}\tbinom{-1}{1}\cdot\mathtt{N}_2)\ \right\} \tag{4.15}$$

is not empty, as well as monotone (Theorem 1.1.12). By Poincaré inequality over Γ (see [43, Th 1.5]),

$$\Big\langle\ k_s\nabla\left(\beta_1-\beta_2\right)\,,\ \nabla\left(\beta_1-\beta_2\right)\ \Big\rangle_\Gamma \geq C\,\|\beta_1-\beta_2\|^2_{H^1(\Gamma)}\,, \tag{4.16}$$

therefore, by Theorem 4.3.1, $\partial\psi = \partial\psi_1 + \partial\psi_2$ is strongly monotone. ■

We want to guarantee the existence of solutions through of Theorem 1.1.25, but not directly since $\partial_W\,\varphi$ is not bounded.

Theorem 4.3.3 (Existence of weak solution) *Suppose* $(\mathtt{u},\mathtt{p},\beta)\in\mathtt{V}\Subset\mathtt{W}$, $\psi:\mathtt{V}\to\,]-\infty,\infty]$, $\varphi:\mathtt{W}\to]-\infty,\infty]$ *as in Definition 4.2.10,* $\varphi_n:\mathtt{W}\to]-\infty,\infty]$ *defined by*

$$\varphi_n(\mathtt{u},\mathtt{p},\beta)_W = \mathtt{R}(\mathtt{z}) + \mathtt{I}^n_{\text{-}}(\beta), \qquad \mathtt{I}^n_{\text{-}}(\beta) = \begin{cases} 0, & \text{if } \beta(\Gamma)\subset\,]-\infty,0] \\ n\,\|\beta^+\|_\Gamma\,, & \text{otherwise}\end{cases},$$

$$\partial_\beta\psi\left(V\right)\subset[0,+\infty[, \qquad F\in\mathtt{V}^*, \quad F(0)-v_0\in D(\varphi_1^*)$$

where $\psi,\varphi,\mathtt{R}$ *are propers, convex, and l.s.c. functions,* $\partial_z\mathtt{R}(\mathtt{z})$ *bounded,* ψ *strongly monotone, and consider the differential inclusions*

$$\partial_W\varphi_n(\dot{\mathtt{u}},\dot{\mathtt{p}},\dot{\beta}) + \partial_V\psi(\mathtt{u},\mathtt{p},\beta) \ \ni\ F, \tag{4.17}$$

$$\partial_W\varphi(\dot{\mathtt{u}},\dot{\mathtt{p}},\dot{\beta}) + \partial_V\psi(\mathtt{u},\mathtt{p},\beta) \ \ni\ F. \tag{4.18}$$

Then the differential inclusion 4.17 has a solution for each $n \in \mathbb{N}$, *and there exists* $m \in \mathbb{N}$ *such that a weak solution* $(\mathtt{u}_n, \mathtt{p}_n, \beta_n)$ *of the differential inclusion 4.17 is also a weak solution of the differential inclusion 4.18 for all* $n \geq m$.

Notice that in this problem $\partial\varphi_n$ is monotone and bounded (Theorem 1.1.12) and by Corollary 4.3.2, $\partial\psi$ is strongly monotone.

Lemma 4.3.4 *If* $F(0) - v_0 \in \partial_W \left\{\mathtt{R}(\mathtt{p}) + \partial\mathtt{I}^k_-(\beta)\right\}_{|(p,\beta)=(0,0)}$, *then for each* $n \geq k \in \mathbb{N}$,

$$F(0) - v_0 \in D(\varphi_n^*).$$

Proof. We will prove that $\langle F(0) - v_0, (\mathtt{u}, \mathtt{p}, \beta)\rangle - \varphi_n(\mathtt{u}, \mathtt{p}, \beta) < \infty$ for all $(\mathtt{u}, \mathtt{p}, \beta) \in \mathtt{W}$. Consider the representation

$$F_u(0) = \begin{pmatrix} F_{0u} \\ F_{0p} \\ F_{0\beta} \end{pmatrix}, \quad v_0 = \begin{pmatrix} v_{0_u} \\ v_{0_p} \\ v_{0_\beta} \end{pmatrix}, \quad \text{where} \quad v_0 \in \partial\psi(\mathtt{u}_0, \mathtt{p}_0, \beta_0).$$

Since $\partial_u\varphi = 0$, $F_u(0) = v_{0_u} = 0$ must be assumed. If $F(0)_p - v_{0_p} \in \partial_{\mathtt{p}}\mathtt{R}(0)$ and $F(0)_\beta - v_{0_\beta} \in \partial\mathtt{I}^k_-(0)$, then for all $(\mathtt{u}, \mathtt{p}, \beta)_W \in \mathtt{W}$,

$$\begin{aligned} \left\langle F(0)_p - v_{0_p}, \mathtt{p}\right\rangle &\leq \mathtt{R}(\mathtt{p}) - \mathtt{R}(0), \\ \left\langle F(0)_\beta - v_{0_\beta}, \beta\right\rangle_\Gamma &\leq \mathtt{I}^k_-(\beta) - \mathtt{I}^k_-(0) \leq \mathtt{I}^n_-(\beta) - \mathtt{I}^n_-(0), \end{aligned}$$

for each $n \geq k$. So

$$\begin{aligned} &\langle F(0) - v_0, (\mathtt{u}, \mathtt{p}, \beta)_W\rangle_W - \varphi_n(\mathtt{u}, \mathtt{p}, \beta)_W \\ &\qquad = \left\langle F(0)_p - v_{0_p}, \mathtt{p}\right\rangle + \left\langle F(0)_\beta - v_{0_\beta}, \beta\right\rangle_\Gamma - \mathtt{R}(\mathtt{p}) - \mathtt{I}^n_-(\beta) \\ &\qquad \leq -\mathtt{R}(0) - \mathtt{I}^n_-(0) < \infty, \end{aligned}$$

and $F(0) - v_0 \in D(\varphi_k^*)$ for all $k \geq n \in \mathbb{N}$. ■

Proof of Theorem 4.3.3. It is enough to check that there exists $m \in \mathbb{N}$ such that for $n \geq m$, $(\mathtt{u}_n, \mathtt{p}_n, \beta_n)$ is a solution of 4.17 and $\dot\beta_n \leq 0$, i.e. $\left(\dot{\mathtt{u}}_n, \dot{\mathtt{p}}_n, \dot\beta_n\right) \in D(\varphi)$ and $\partial\mathtt{I}^n_-(\dot\beta_n) \subset \partial\mathtt{I}_-(\dot\beta_n)$. In fact, $\partial\varphi_n$ is monotone and bounded, and $\partial\psi$ is strongly monotone. On the other hand, $F(0) - v_0 \in D(\varphi_n^*)$ for all $n \geq k \in \mathbb{N}$ (Lemma 4.3.4), so by Theorem 1.1.25, for each $n \geq k$ there exist $(\mathtt{u}_n, \mathtt{p}_n, \beta_n) \in H^1(0, T; \mathtt{V})$, $w_n \in L^\infty(0, T; \mathtt{W}^*)$,

$v_n \in L^1(0,T;\mathtt{W}^*) \cap L^\infty(0,T;\mathtt{V}^*)$ that satisfy differential inclusion 4.17, this is,

$$\begin{array}{rcl} w_n(t) + v_n(t) & = & F(t), \\ w_n(t) & \in & \partial\varphi_n(\dot{\mathtt{u}}_n, \dot{\mathtt{p}}_n, \dot{\beta}_n)(t), \\ v_n(t) & \in & \partial\psi_n(\mathtt{u}_n, \mathtt{p}_n, \beta_n)(t) \qquad for \quad a.e. \quad t \in]0,T[, \\ (\mathtt{u}_n, \mathtt{p}_n, \beta_n)(0) & = & (\mathtt{u}_{0n}, \mathtt{p}_{0n}, \beta_{0n}). \end{array}$$

therefore it remains to prove that for some $m \in \mathbb{N}$, $\dot{\beta}_n \leq 0$, for all $n \geq m$. Suppose the opposite, that is, there exists a sub-succession β_{n_i} of β_n such that $\dot{\beta}^+_{n_i} > 0$ for all $i \in \mathbb{N}$. Let $F = w_{n_i} + v_{n_i} \in \partial_W\varphi\left(\dot{\mathtt{u}}_{n_i}, \dot{\mathtt{p}}_{n_i}, \dot{\beta}_{n_i}\right) + \partial_V\psi\left(\mathtt{u}_{n_i}, \mathtt{p}_{n_i}, \beta_{n_i}\right)$, i.e. $F := \left(F_{u_{n_i}}, F_{p_{n_i}}, F_{\beta_{n_i}}\right)$, where $F_{\beta_{n_i}} \in \partial_\beta \mathtt{I}^n_-(\dot{\beta}^+_{n_i}) + \partial_\beta\psi\left(\mathtt{u}_{n_i}, \mathtt{p}_{n_i}, \beta_{n_i}\right)$, $\partial\mathtt{I}^n_-(\dot{\beta}^+_{n_i}) = \frac{n_i}{\left\|\dot{\beta}^+_{n_i}\right\|_\Gamma}\dot{\beta}^+_{n_i}$ and $\varsigma_{n_i} \in \partial_\beta\psi\left(\mathtt{u}_{n_i}, \mathtt{p}_{n_i}, \beta_{n_i}\right)$. Realizing the product between F and $(0,0,\dot{\beta}^+_{n_i})$,

$$\begin{array}{rcl} \langle F, (0,0,\dot{\beta}^+_{n_i})\rangle_W & = & \langle \left(F_{u_{n_i}}, F_{p_{n_i}}, F_{\beta_{n_i}}\right), (0,0,\dot{\beta}^+_{n_i})\rangle_W \\ & = & \langle F_{\beta_{n_i}}, \dot{\beta}^+_{n_i}\rangle_\Gamma \\ & = & \langle \frac{n_i}{\left\|\dot{\beta}^+_{n_i}\right\|_\Gamma}\dot{\beta}^+_{n_i}, \dot{\beta}^+_{n_i}\rangle_\Gamma + \langle \varsigma_{n_i}, \dot{\beta}^+_{n_i}\rangle_\Gamma \\ & \geq & \frac{n_i}{\left\|\dot{\beta}^+_{n_i}\right\|_\Gamma}\langle \dot{\beta}^+_{n_i}, \dot{\beta}^+_{n_i}\rangle_\Gamma . \end{array}$$

That is,

$$\|F\|_V \geq \langle F_{\beta_{n_i}}, \dot{\beta}^+_{n_i} / \left\|\dot{\beta}^+_{n_i}\right\|_\Gamma \rangle \underset{n_i\to\infty}{\longrightarrow} \infty,$$

which is absurd. Therefore, there exists $m \in \mathbb{N}$ such that $\dot{\beta}_n \leq 0$ for $n \geq m$, and $(\mathtt{u}_m, \mathtt{p}_m, \beta_m)$ is a solution of the differential inclusion 4.18. ■

Chapter 5

Conclusions and Perspectives.

Two transmission problems and two adhesive contact problems were addressed in this thesis. For the transmission problem of perfectly plastic plates, a formulation was obtained in the form of a nonlinear Cauchy problem, and it was proved that under certain boundary conditions, the problem is well-posed.

In the problems of adhesion with elastoplasticity of solids and plates, weak formulations were obtained and the existence of such solutions was proved. Each of the developed models can be expanded if other physical aspects such as temperature, damage and friction are considered.

The model formulations do not omit details about the contact region, and the spaces for the displacements are considered in a Sobolev space $W^{k,p}$, with $k \geq 1$, in this way, the Trace theorem guarantees to extend the displacements defined in the interior of the solids to its boundary.

The construction of the theorems, auxiliary lemmas, and their proofs are original and relatively simple because they resort to established results in the field of monotone PDEs. Readers can interpret the results without any prior knowledge of the calculus of variations. However, there are still contributions to be developed, particularly in the problems of uniqueness and regularity of solutions, and the numerical approach of the models.

Considering unidirectional adhesion in viscoelastic and perfectly plastic models (i.e. rate dependent processes) under this approach requires formulating doubly nonlinear problems with time-dependent operators. The reason is that the differential inclusions for adhesion are defined in terms of the displacements in the contact zone, while the momentum equations are defined in terms of the velocities. By coupling these equations, the displacements and therefore the operators are expressed from integrals of the velocity as a function of t.

An updated list of completed developments and doubly nonlinear problems still to be solved is presented in Table 1. From a completely analysis, it is suggested to address those problems to be solved ($\times$).

Problem	Autonomous	E U	Non-autonomous	E U
Cauchy nonlinear	$\frac{du}{dt}+Au \ni f$	$\checkmark\checkmark$	$\frac{du}{dt}+A(t)u \ni f$	$\checkmark\checkmark$
Doubly nonlinear	$B(\frac{du}{dt}) + Au \ni f$	$\checkmark\times$	$B(\frac{du}{dt}) + A(t)u \ni f$	$\times\times$
Doubly nonlinear	$\alpha(\frac{du}{dt})$-div$(\beta\nabla u) \ni f$	$\checkmark\checkmark$	$B(t)(\frac{du}{dt}) + Au \ni f$	$\times\times$
Doubly nonlinear	$\frac{d}{dt}(Bu) + Au = f$	$\checkmark\checkmark$	$B(t)(\frac{du}{dt})+A(t)u \ni f$	$\times\times$
		E:	Existence of solutions	
		U:	Uniqueness of solutions	

Table 1: Completed and unfinished problems in the theory of monotone operators on Banach spaces.

In particular, the formulation of the perfectly plastic plate transmission problem was represented in the form of the nonlinear Cauchy problem $\frac{du}{dt}+Au \ni f$, while the adhesive contact and elastoplastic deformation models were weakly represented as $B(\frac{du}{dt}) + Au \ni f$.

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